REV-01
MSM/02/07

2024/11

SET A

M.Sc. MATHEMATICS
FIRST SEMESTER
REAL ANALYSIS
MSM – 101 [REPEAT]
[USE OMR FOR OBJECTIVE PART]

Duration: 3 hrs. **Full Marks: 70**

Objective

Time: 30 min. **Marks: 20**

Choose the correct answer from the following: ***1×20=20***

1. A series of functions $\sum f_n$ will converge uniformly on $[a, b]$ if there is a convergent series $\sum M_n$ of positive numbers such that for all $x \in [a, b]$ and for all n
   a. $|f_n(x)| < M_n$
   b. $|f_n(x)| > M_n$
   c. $|f_n(x)| \leq M_n$
   d. $|f_n(x)| \geq M_n$

2. Show that the sequence $< f_n >$, where $f_n(x) = x^n$ is uniformly convergent to the limit function $f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$ in the interval
   a. $[0, 1]$
   b. $[0, k], k < 1$
   c. $[0, k], \quad k > 1$
   d. $[-1, 1]$

3. $\sum \frac{\cos n\theta}{n^p}$ is uniformly convergent for all real values of θ for
   a. $p \leq 1$
   b. $p \geq 1$
   c. $p > 1$
   d. $p < 1$

4. The statement that any uniformly convergent series of functions is also pointwise convergent but the converse is not necessarily true is
   a. False
   b. True
   c. Neither false nor true
   d. Only conditionally true

5. The power series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ is
   a. Nowhere convergent
   b. Everywhere convergent
   c. Convergent at $x = 1$ only
   d. Convergent at $x = \frac{1}{2}$

6. The radius of convergent R for a power series $\sum_{n=0}^{\infty} a_n x^n$ is computed by the formula
   a. $R = \overline{\lim}\ |a_n|$
   b. $R = \overline{\lim}\,|a_n|^{1/n}$
   c. $R = \dfrac{1}{\overline{\lim}\,|a_n|^{1/n}}$
   d. None of these

7. If a power series $\sum a_n x^n$ converges for $x = x_0$, then it is absolutely convergent for $x = x_1$ where
   a. $x_1 < x_0$
   b. $|x_1| < |x_0|$
   c. $x_1 < |x_0|$
   d. $|x_1| < x_0$

8. The series $\sum \frac{x^n}{n^2}$ is uniformly convergent in
   a. $[-1, 1]$
   b. $(-1, 1)$
   c. $(-1, 1]$
   d. $[-1, 1)$

9. The radius of convergence of the series $1 + 2x + 3x^2 + 4x^3 + \cdots$

a. $\frac{1}{2}$
b. 1
c. $\frac{3}{2}$
d. None of these

10. The radius of convergence of the series $\sum_{n=1}^{\prime} \frac{x^n}{n^n}$ is

a. 0
b. ∞
c. 1
d. None of these

11. Consider $\mathbb{R}$, the set of all real numbers with discrete metric $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$ where $x, y \in \mathbb{R}$. Then $d(1,3)$ is equal to

a. -2
b. 2
c. 0
d. 1

12. Consider $\mathbb{R}^2 = \{x = (x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$. For any $x, y \in \mathbb{R}^2$ with $x = (x_1, x_2)$, $y = (y_1, y_2)$ define $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$. If $x = (-1, 1)$, $y = (2, -1)$ then $d(x,y)$ is

a. 4
b. 5
c. 6
d. 7

13. Define $d_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $d_1(x,y) = \text{Max}\{|x_1 - y_1|, |x_2 - y_2|\}$. If x, y are exactly same as question 12, then $d_1(x,y)$ will be

a. 2
b. 3
c. 1
d. 4

14. For any two points $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ define $d(x,y) = |x_1 - y_1|$. Then

a. $d(x,y) = 0 \Rightarrow x = y$
b. $x = y \Rightarrow d(x,y) = 0$
c. $d(x,y) = 0 \Leftrightarrow x = y$
d. None of these

15. Consider $\mathbb{R}$, the set of real numbers with discrete metric d on $\mathbb{R}$ defined by $d(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$. If $A = \{1, 2, 4, 7\}$ then $d(10, A)$ is

a. 1
b. 2
c. 3
d. 4

16. Consider $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$ with usual metric d as defined by $d(x,y) = |x - y|, x, y \in \mathbb{N}$. If $A = \{2, 4, 5\}$ then $d(A)$ is

a. 0
b. 2
c. 1
d. 3

17. Consider $\mathbb{R}$, the set of all real numbers with usual metric d. Then $S_2(1)$, the open sphere with centre 1 and radius 2 is

a. $-1 \leq x < 3$
b. $-1 < x \leq 3$
c. $-1 < x < 3$
d. $-1 \leq x \leq 3$

18. If (X, d) is a metric space with discrete metric d, then

a. No subset of X is open
b. Every subset of X is open
c. ϕ and X are only open subsets of X
d. None of these is true

19. In any metric space (X, d)

a. Union of open sets may not be open

b. Union of only finite number of open sets is open.

c. Union of any number of open sets is open.

d. None of these is true.

20. In any metric space (X, d)

a. Intersection of open sets may not be open

b. Intersection of any finite number of open sets is open

c. Intersection of any number of open sets is open

d. None of these is true

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Descriptive

Time : 2 hrs. 30 min. Marks:50

[Answer question no.1 & any four (4) from the rest]

1. a. Define a metric space (X, d). Explain the concepts of 2+3+2+3 =10
(i) discrete metric on $\mathbb{R}$
(ii) usual metric on $\mathbb{R}$
(iii) usual metric on $\mathbb{R}^2$

b. What is a power series in x ? Explain the following terms with examples-
(i) Radius of convergence of a power series
(ii) Nowhere convergent power series
(iii) Everywhere convergent power series

2. a. Let (X, d) be a metric space. Define $d_1 : X \times X \to \mathbb{R}$ by 5+5=10

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}, x, y \in X$$

Examine if (X, d_1) is also a metric space or not.

b. What do you mean by an open sphere $S_r(x_0)$ with centre x_0 and radius r in a metric space (X, d).Find $S_{\frac{1}{2}}(0)$ in $\mathbb{R}$ with respect to both usual metric and discrete metric.

3. a. When is a set G said to be open in a metric space (X, d)? Consider $G = \{1, 2, 3\}$. Show that G is open in $\mathbb{R}$ with discrete metric, but it is not open in the usual metric on $\mathbb{R}$. 5+5=10

b. Consider the sequence of function $< f_n(x) >$, where $f_n(x) = x^n$. Show that $< f_n(x) >$ converges pointwise for $-1 < x \leq 1$. Find also the limit function f for the given sequence.

4. a. State and prove Weierstrass's M-test for uniform convergence of a series of functions $\sum f_n(x)$ in $[a, b]$. 5+5=10

b. Apply Weierstrass's M-test to show that $\sum x^n \cos n\theta$ converges uniformly for all real values of θ if $0 < x < 1$.

5. a. Test the uniform convergence of the sequence of functions $< f_n(x) >$ where $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$. 5+5=10

b. If a sequence $< f_n(x) >$ converges uniformly to f in $[a, b]$ and $x_0 \in [a, b]$ such that $\lim_{x \to x_0} f_n(x) = a_n$, $n = 1, 2, 3, \cdots$ then prove that $< a_n >$ converges and $\lim_{x \to x_0} f(x) = \lim_{n \to \infty} a_n$

6. a. Find the radius of convergence of the power series 5+5=10

$$\frac{1}{2}x + \frac{1}{2} \cdot \frac{3}{5}x^2 + \frac{1}{2} \cdot \frac{3}{5} \cdot \frac{5}{8}x^3 + \cdots$$

b. If a power series $\sum a_n x^n$ converges for $x = x_0$ then prove that it is absolutely convergent for every $x = x_1$ such that $|x_1| < |x_0|$

7. a. If a power series $\sum a_n x^n$ converges for $|x| < R$ and there is a function $f(x)$ with 5+5=10

$$f(x) = \sum a_n x^n |x| < R$$

then prove that $\sum a_n x^n$ converges uniformly on $[-R + \epsilon, R - \epsilon]$ and that $f(x)$ is continuous and differentiable on $(-R, R)$ with

$$f'(x) = \sum n a_n x^{n-1}, |x| < R$$

b. Show that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$, $-1 \leq x \leq 1$ and hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

8. a. For $x = (x_1, x_2), y = (y_1, y_2)$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$let us define distance d between x and y by $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$. Show that $(\mathbb{R}^2, d)$ is a metric space. 5+5=10

b. Show that any open sphere in a metric space (X, d) is an open set.

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