2024/05

SET A

M.SC. MATHEMATICS
SECOND SEMESTER
TOPOLOGY & FUNCTIONAL ANALYSIS
MSM – 202

[USE OMR FOR OBJECTIVE PART]

Duration: 3 hrs. **Full Marks: 70**

(Objective)

Time: 30 min. **Marks: 20**

Choose the correct answer from the following: ***1×20=20***

1. Consider the following collection on $X = \{a, b, c, d, e\}$

$$\tau_1 = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$$
$$\tau_2 = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$
$$\tau_3 = \{\phi, X, \{a\}, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}\}$$

   a. τ_1, τ_2 and τ_3 all are topology on X.
   b. Only τ_2 and τ_3 are topology on X.
   c. Only τ_1 is topology on X.
   d. None of the above.

2. Let $\mathfrak{J} = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ be a topology on $X = \{a, b, c, d, e\}$ and let $A = \{a, d, e\}$. Then $\mathfrak{J}_A$ is equal to
   a. $\{\phi, A, \{a\}, \{d\}, \{a, d\}, \{d, e\}\}$
   b. $\{\phi, A, \{a\}, \{d\}, \{c, d\}, \{a, e\}\}$
   c. $\{\phi, A, \{a\}, \{d\}, \{a, d\}, \{c, d\}\}$
   d. None of these

3. Let $(X, \mathcal{T}_d)$ be a discrete topological space. Then which of the followings is/are true for $\mathcal{T}_d$
   a. $\{\{x\} : \text{for some } x \in X\}$ is a base
   b. $\{\{x\} : \forall x \in X\}$ is not a base
   c. $\{\{x\} : \forall x \in X\}$ is a base
   d. None of these

4. Let $\mathcal{T}$ denotes the usual topology on the real line $\mathbb{R}$. The subspace topology $\mathcal{T}_{\mathbb{N}}$ defined on the set of natural number is
   a. Equal to the discrete topology
   b. Equal to the indiscrete topology
   c. Neither discrete nor indiscrete
   d. Can not define

5. Which of the following topological space is first countable?
   a. Indiscrete Topology on $\mathbb{R}$
   b. Usual Topology on $\mathbb{R}$
   c. Both (a) and (b) are true
   d. Neither (a) nor (b) is true

6. Which of the following is/are always true?
   a. Discrete topological space $(X, \mathcal{D})$ is always first countable.
   b. Discrete topological space $(X, \mathcal{D})$ is first countable iff X is finite.
   c. Discrete topological space $(X, \mathcal{D})$ is first countable iff X is countable.
   d. Discrete topological space $(X, \mathcal{D})$ is not first countable.

7. Which of the following is/are true?
   a. Discrete topological space is compact but not connected.
   b. Discrete topological space is connected but not compact.
   c. Discrete topological space is both compact and connected.
   d. Discrete topological space is neither compact nor connected.

8. Which of the following is/are true?
   a. Discrete topological space is T_2-space.
   b. Co-finite topological space is T_2-space
   c. Real line with usual topology is T_1-space but not T_2.
   d. None of these

9. If $X = \{a, b, c, d\}$ then which of the following topology is/are connected?
   a. $\tau = p(X)$
   b. $\tau = \{\phi, X, \{a\}\}$
   c. $\tau = \{\phi, \{a, b\}, \{c, d\}, X\}$
   d. None of these.

10. In a normed space $(X, \|\cdot\|)$
    a. All subsets are both open and closed.
    b. No subset of X is both open and closed.
    c. Subsets of X which are both open and closed are the empty set ϕ and the whole space X only
    d. None of these.

11. For a normed linear space X over the field $\mathbb{R}$ of real numbers consider the following subspaces

    B : the subspace of all bounded sequences in X

    C : the subspace of all convergent sequences in X

    C_0: the subspace of all sequences in X converging to zero.

    Then
    a. B is a subspace of C
    b. C is a subspace of C_0
    c. B is a subspace of C_0
    d. C_0 is a subspace of C

12. Let X be a normed space over the field F of real or complex numbers. Let M be a closed subspace of X so that under a suitable norm the quotient space $\frac{X}{M}$ is also a normed space over F. Then $\frac{X}{M}$ is complete if
    a. X is complete.
    b. M is complete as a subspace of X
    c. X is convex in X
    d. Both (b) and (c) are true.

13. In a normed space $(X, \|\cdot\|)$ for any $x, y \in X$, $|\| x \| - \| y \||$ is
    a. less than $\| x - y \|$
    b. equal to $\| x - y \|$
    c. greater than $\| x - y \|$
    d. Both (a) and (b) are true

14. Let ℓ_2^p denote the set of all ordered pairs of real numbers of the form $x = (x_1, x_2)$, $x_1, x_2 \in \mathbb{R}$. A norm $\|\cdot\|_p$ is defined on ℓ_2^p as $\| x \|_p = (|x_1|^p + |x_2|^p)^{\frac{1}{p}}$. Then $(\ell_2^p, \|\cdot\|_p)$ is a normed space if
    a. $0 < p < 1$
    b. $0 < p \leq 1$
    c. $0 < p < \infty$
    d. $1 \leq p < \infty$

15. Let $B(X,Y)$ denote the linear space of all bounded linear operators from a normed space X into another normed space Y. If $\|\cdot\|$ is defined on $B(X,Y)$ by

$$\| T \| = \text{Sup}\{\| T(x) \|_Y : x \in X, \| x \|_X \leq 1\}$$

then $(B(X,Y), \|\cdot\|)$ is Banach space if

a. X is a Banach space
b. Y is a Banach space
c. Both X and Y are Banach space
d. None of these

16. Two norms $\|\cdot\|$ and $\|\cdot\|'$ on the same linear space X will be equivalent if the identity map $I : (X, \|\cdot\|) \to (X, \|\cdot\|')$ is

a. isometric
b. an isomorphism
c. a homeomorphism
d. both isomorphism and homeomorphism

17. Let X and Y be two normed spaces of dimensions m and n respectively. X is topologically isomorphic to Y if

a. $m < n$
b. $m = n$
c. $m > n$
d. None of the above

18. Let X and Y be normed spaces over the same field $F(= \mathbb{R}$ or $\mathbb{C})$. Let $T : X \to Y$ be a linear operator with the property

(i) T is bounded
(ii) T maps every bounded set in X onto bounded set in Y

then

a. $(i) \Rightarrow (ii)$
b. $(ii) \Rightarrow (i)$
c. Both (a) and (b) hold
d. None of (a) and (b) holds

19. Let X and Y be normed spaces over the same filed $F(= \mathbb{R}$ or $\mathbb{C})$. Let $T: X \to Y$ be a linear operator with the property

(i) T is continuous at the origin
(ii) T is continuous on X

then

a. $(i) \Rightarrow (ii)$
b. $(ii) \Rightarrow (i)$
c. Both (a) and (b) hold
d. None (a) and (b) holds

20. Which of the following is/are open cover of discrete topological space $(X, \mathfrak{D})$, where $X = \{a, b, c\}$?

a. $\{\{a\}, \{b\}, \{c\}\}$
b. $\{\{a\}, \{b\}, \{c\}, \{a, b\}\}$
c. $\{\phi, X, \{a\}\}$
d. All the above

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(Descriptive)

Time : 2 hrs. 30 mins. Marks : 50

[Answer question no.1 & any four (4) from the rest]

1. a. Define Continuity of a function f from a topological space (X,τ) into another topological space (Y,τ^*) . Consider(X,τ) and (Y,τ^*) where 1+2+2+5=10

$$X = \{a,b,c,d\}, \tau = \{X,\phi,\{a\},\{a,b\},\{a,b,c\}\}$$

and $Y = \{p,q,r,s\}, \tau^* = \{Y,\phi,\{p\},\{q\},\{p,q\},\{q,r,s\}\}$

Examine if the function $f : X \to Y$ and $g : X \to Y$ as defined below are continuous or not

$$f(a) = q, f(b) = r, f(c) = s, f(d) = r$$

and $g(a) = p, g(b) = p, g(c) = r, g(d) = s$

b. Let p be a real number such that $1 \leq p < \infty$. Consider the normed space ℓ_n^p of all n-tuples $x = (x_1, x_2, \cdots, x_n)$ of scalars under the norm defined by $\| x \|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ Show that $(\ell_n^p, \|\cdot\|_p)$ is a Banach space.

2. a. Let X be a set and let $\tau_f = \{U \subseteq X : X - U$ is either finite or X$\}$. Prove that (X,τ_f) is a topological space. 4+4+2=10

b. Show that - $A = (0,1) \subseteq \mathbb{R}$ with usual topology is not compact.

c. Show that- The real line $\mathbb{R}$ with lower limit topology τ_ℓ is not connected space.

3. a. Show that the lower limit topology on $\mathbb{R}$ is finer than the usual topology on $\mathbb{R}$. Also, show that the lower limit topological space is not second countable. 5+5=10

b. Let (X,τ_1) and (Y,τ_2) be two topological spaces. Prove that – If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.Is the result true for order topology and subspace topology same? Justify your answer.

4. Prove that – For a topological space (X, τ) the following statements are equivalent: **10**

(i) The space X is a T_1-space.
(ii) For any $x \in X$, the singleton set $\{x\}$ is closed.
(iii) Every finite subset of X is closed.
(iv) The topology τ is stronger than the co-finite topology on X.

5. **a.** Construct an open cover of a topological space (X, τ), where $X = \{a, b, c, d\}$ and **3+2+5 =10**
(i) $\tau = p(X)$
(ii) $\tau = \{\phi, X\}$
(iii) $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, X\}$

b. Show that – $\mathcal{A} = \{(-n, n) : n \in \mathbb{N}\}$ is an open cover of the real line with usual topology. Find an open subcover for $\mathcal{A}$.

c. Let X be any normed vector space over a field F. For any $x, y \in X$, show that $|\| x \| - \| y \|| \leq \| x - y \|$.

6. **a.** Let $(X, \|\cdot\|)$ be a normed space over a field F where F is $\mathbb{R}$ or $\mathbb{C}$. If $u_n \to u$ and $\lambda_n \to \lambda$ then prove that $\lambda_n u_n \to \lambda u$ as $n \to \infty$. **5+5=10**

b. Prove that a normed space $(X, \|\cdot\|)$ is homeomorphic to the open ball $\{x \in X : \| x \| < 1\}$.

7. **a.** Let Y be a subspace of a Banach space X. If Y is closed then prove that Y is complete Y is also a Banach space. **5+5=10**

b. When is a linear operator $T : X \to Y$ from a normed space X to another normed space Y said to be bounded?
Prove that a linear operator $T : X \to Y$ from a normed space X into a normed space Y is continuous if and only if T is bounded.

8. **a.** For a normed space X let $B(X)$ denote the set of all bounded linear operators on X. If $S, T \in B(X)$. then S T ∈ B(x) **5+5=10**
Also, establish the inequality

$$\| ST \| \leq \| S \| \| T \|$$

b. When are two normed spaces said to be topologically isomorphic?
If X and Y are two normed spaces over a filed $F(= \mathbb{R}$ or $\mathbb{C})$ then a linear operator $T : X \to Y$ will be a topological isomorphism if and only if there exist constants $k_1, k_2 > 0$ such that

$$k_1 \parallel x \parallel_X \leq \parallel T(x) \parallel_Y \leq k_2 \parallel x \parallel_X \quad \forall x \in X$$

Where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are respectively norms on X and Y.

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