REV-01
MSM/01/05

2024/07

SET A

M.Sc. MATHEMATICS
FIRST SEMESTER
REAL ANALYSIS
MSM – 101 [SPECIAL REPEAT]

[USE OMR FOR OBJECTIVE PART]

Duration: 3 hrs. **Full Marks: 70**

(**Objective**)

Time: 30 min. **Marks: 20**

Choose the correct answer from the following: ***1X20=20***

1. Let $S(x, r)$ be an open sphere in a discrete metric space $(X, \mathcal{D})$. Then $S(x, r)$ is a singleton set if
   a. $0 < r < 1$
   b. $0 < r \leq 1$
   c. $r \geq 1$
   d. $r > 1$

2. Consider $\mathbb{R}$, the set of real numbers with usual metric d on $\mathbb{R}$ given by $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. Then $S\left(-1, \frac{3}{2}\right)$ is equal to
   a. $\left]-\frac{5}{2}, \frac{1}{2}\right]$
   b. $\left[-\frac{5}{2}, \frac{1}{2}\right[$
   c. $\left]-\frac{5}{2}, \frac{1}{2}\right[$
   d. $\left[-\frac{5}{2}, \frac{1}{2}\right]$

3. Let (X, d) be any metric space, and $A \subset X$. Then the interior of A is the
   a. Intersection of all open sets contained in A.
   b. Intersection of all open sets containing A.
   c. Union of all open sets containing A.
   d. Union of all open sets contained in A.

4. Let (X, d) be any metric space, and $A \subset X$. Then the closure of A is the
   a. Intersection of all closed sets contained in A.
   b. Union of all closed sets contained A.
   c. Intersection of all closed sets containing A.
   d. Union of all closed sets containing in A.

5. Let $< x_n >$ be any sequence in a metric space (X, d). If $< x_n >$ converges then
   a. the sequence is Cauchy
   b. the sequence is not Cauchy
   c. the sequence is not bounded
   d. None of these is true

6. Let $\sum f_n(x)$ be a series of continuous functions defined on $[a, b]$ for each n, converging pointwise to the sum function f. Then
   a. f is continuous on $[a, b]$
   b. f is discontinuous on some point in $[a, b]$
   c. f may or may not be continuous on $[a, b]$
   d. None of these

7. Let a sequence $\{f_n\}$ of real functions converges uniformly to a real function f so that given $\epsilon > 0$, there exists a positive integer m so that $|f_n(x) - f(x)| < \epsilon,\ \forall n \geq m$ for $x \in [a, b]$. Then

a. m depends on $x \in [a, b]$ and not on ϵ

b. m depends on ϵ and not on $x \in [a, b]$

c. m is independent of both ϵ and $x \in [a, b]$

d. None of these

8. Let $< f_n >$ be a sequence of functions such that $\lim_{n\to\infty} f_n(x) = f(x), \forall x \in [a, b]$ and let $M_n = \text{Sup}_{x\in[a,b]}|f_n(x) - f(x)|$

a. $M_n \to +\infty$ as $n \to \infty$

b. $M_n \to 0$ as $n \to \infty$

c. $M_n \to -\infty$ as $n \to \infty$

d. M_n is bounded for all n

9. The sequence $< f_n >$ of functions where $f_n(x) = x^n$ defined on $[0, 1]$ is convergent to the limit function f where

a. $f(x) = 1,\ \forall x \in [0, 1]$

b. $f(x) = 0,\ \forall x \in [0, 1]$

c. $f(x) = \begin{cases} 1, & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0 \end{cases}$

d. $f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

10. Consider the series $\sum f_n$ of functions where $f_n(x) = \frac{x^2}{(1+x^2)^n}$, $x \in \mathbb{R}$. The series converges to a sum function f given by

a. $f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$

b. $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$

c. $f(x) = 0,\ x \in \mathbb{R}$

d. $f(x) = 1, \quad x \in \mathbb{R}$

11. For any interval $[a, b]$ in $\mathbb{R}$ the length of $[a, b]$ is

a. $a + b$

b. $a - b$

c. $b - a$

d. None of the above

12. If G is any open set in $\mathbb{R}$ then

a. G is union of a countable class of open intervals.

b. G is union of a disjoint class of open intervals

c. G is union of a countable disjoint class of open intervals

d. None of the above

13. For any set $A \subseteq [a, b]$, the outer measure m^*A is defined by

a. Sup $l(F)$, where the supremum is taken over the length of all open sets $F \supseteq A$.

b. Inf $l(F)$, where the infimum is taken over the length of all open sets $F \supseteq A$.

c. Inf $l(F)$, where the infimum is taken over the length of all open sets $F \subseteq A$.

d. None of these

14. For any two subsets A_1 and A_2 in $[a, b]$

a. $m^*A_1 + m^*A_2 \leq m^*(A_1 \cup A_2) + m^*(A_1 \cap A_2)$

b. $m^*A_1 + m^*A_2 \geq m^*(A_1 \cup A_2) + m^*(A_1 \cap A_2)$

c. $m_*A_1 + m_*A_2 \geq m_*(A_1 \cup A_2) + m_*(A_1 \cap A_2)$

d. None of these

15. If A be any subset of $[a, b]$ and m_*A is the inner measure of A then given $\mathcal{E} > 0$, there is a closed set $G \subset A$ such that

a. $m_*A - \mathcal{E} < l(G)$
b. $m_*A + \mathcal{E} < l(G)$
c. $m_*A - \mathcal{E} > l(G)$
d. None if these

16. The radius of convergence of the power series $1 + 2x + 3x^2 + 4x^3 + \cdots$ is

a. 0
b. $\frac{1}{2}$
c. 1
d. 2

17. The power series $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ is

a. Convergent at $x = 0$ only.
b. Everywhere Convergent
c. Nowhere Convergent
d. None of these

18. The radius of convergence R of a power series $\sum a_n x^n$ is given by

a. $R = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$
b. $R = \frac{1}{\lim_{n\to\infty} |a_n|^{\frac{1}{n}}}$
c. $R = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|^{\frac{1}{n}}$
d. $R = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

19. The interval of convergence of the power series $1 + x^2 + x^4 + x^6 + \cdots$ is

a. $-1 \le x < 1$
b. $-1 < x \le 1$
c. $-1 < x < 1$
d. $-1 \le x \le 1$

20. The radius of convergence of the power series $x + \frac{x^2}{2^2} + \frac{2!}{3^3}x^3 + \frac{3!}{4^4}x^4 + \cdots$

a. $\frac{1}{e}$
b. e
c. $1 + e$
d. $1 - e$

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Descriptive

Time : 2 hrs. 30 min. Marks : 50

[Answer question no.1 & any four (4) from the rest]

1\. a. Let (X, d) be any metric space. Define a metric d_1 on X by 5+2+1+2=10

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}, x, y \in X$$

Show that (X, d_1) is again a metric space.

b. Consider the sequence of the functions $< f_n >$, where

$$f_n(x) = \frac{\sin nx}{\sqrt{x}}, x \in \mathbb{R}$$

Is $< f_n >$ convergent? If so, find the limit function f for $< f_n >$.

Examine the convergence of $< f_n' >$, where $f_n'(x) = \frac{d}{dx} f_n(x)$, $x \in \mathbb{R}$.

2\. a. When is a sequence $< x_n >$ said to be convergent in a metric space (X, d)? Prove that a convergent sequence in a metric space is always Cauchy. 1+3+1+3+2=10

Give an example to show that a Cauchy sequence in a metric space (X, d) may not be convergent.

b. Consider the series of functions $\sum f_n$, where $f_n(x) = \frac{x^2}{(1+x^2)^n}$, $x \in \mathbb{R}$

Examine the convergence of $\sum f_n$ and find the sum function f provided $\sum f_n$ is convergent. What is your observation on continuity of each term f_n and that of the sum function f?

3\. a. Prove Cauchy's criterion for uniform convergence of a series of functions $\sum f_n$ viz- 5+2+1+2=10

A series of functions $\sum f_n$ defined on an interval $I = [a, b]$ converges uniformly if and only if for $\mathcal{E} > 0$, and for all $x \in [a, b]$, there exists a positive integer m such that

$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)| < E, \forall n \geq m,, p \geq 1$$

b. Examine the convergence of the sequence of function $< f_n >$ where $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$

Find the limit function f in case it is convergent. Also, in this case establish whether the convergence of the sequence is pointwise or uniform.

4. a. Prove Weierstrass's M-test viz – 5+5=10

A series of function $\sum f_n$ will converge uniformly and absolutely on $[a, b]$ if there is a convergent series $\sum M_n$ of positive numbers such that

for all $x \in [a, b]$, $|f_n(x)| \leq M_n \quad \forall n$.

b. Show that the series $\sum \frac{x}{n^p + x^2 n^q}$ converges uniformly over any finite interval $[a, b]$ for $0 < p \leq 1,\ p + q > 2$.

5. a. Let $\sum f_n$ be a sequence of functions converging uniformly to a limit function f in interval $[a, b]$. If f_n is continuous for each n in $[a, b]$, then prove that the limit f is also continuous in $[a, b]$. 5+5=10

b. Show that the series $\sum f_n$, where $f_n(x) = \frac{x^4}{(1+x^4)^{n-1}}$ is not uniformly continuous though it is pointwise convergent in $[0, 1]$.

6. a. Define radius of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$. Write a formula to find the radius of convergence R for $\sum a_n x^n$. Hence find the radius of convergence for the power series 1+1+3+5=10

$$1 + 2x + 3x^2 + 4x^3 + \cdots$$

b. If a power series $\sum a_n x^n$ converges for $x = x_0$ then prove that it is absolutely convergent for every $x = x_1$ where $|x_1| < |x_0|$

7. a. Prove Abel's theorem on uniform convergence of a power series $\sum a_n x^n$ viz – 5+2+3 =10

If a power series $\sum a_n x^n$ converges at end point $x = R$ of the interval $]-R, R[$ then it is uniformly convergent in the closed interval $[0, R]$.

b. Show that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, -1 \leq x \leq 1$. Also show that $\frac{1}{2}(\tan^{-1} x)^2 = \frac{x^2}{2} - \left(1 + \frac{1}{3}\right)\frac{x^4}{4} + \left(1 + \frac{1}{3} + \frac{1}{5}\right)\frac{x^6}{6} - \cdots, -1 \leq x \leq 1$

8. **a.** Define outer measure and inner measure of a set $A \subset [a, b]$. Hence show that $m_* A \leq m^* A$ — 2+3+2+2+1=10

b. Prove that - If A_1 and A_2 are measurable sets in $[a, b]$ then both $A_1 \cup A_2$ and $A_1 \cap A_2$ are also measurable and

$$mA_1 + mA_2 = m(A_1 \cup A_2) + m(A_1 \cap A_2)$$

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