$$= 1 + 2zA(z) + \sum_{n=1}^{\infty} z^n$$

$$= 1 + 2zA(z) + \left(\frac{1}{1 - z} - 1\right)$$

Simple algebraic manipulation yields

$$A(z) = \frac{1}{(1 - z)(1 - 2z)}$$

$$= \frac{2}{1 - 2z} - \frac{1}{1 - z} \quad \text{(breaking into partial fractions)}$$

The corresponding sequence is given by $t_n = 2^{n+1} - 1$, since $1/(1 - 2z)$ generates the sequence 1, 2, 2^2, ... and $1/(1 - z)$ generates the sequence 1, 1, 1, ...

Note that initially, we had a problem to be solved for t_n and we converted it into a simpler problem to be solved for A. In tackling a problem, we can choose a domain that makes the problem easier to solve.

EXAMPLE 1.19: Consider the recurrence relation

$$t_n - 5t_{n-1} + 6t_{n-2} = 2^n + n,\ n \geq 2 \text{ with } t_0 = 1 \text{ and } t_1 = 1.$$

Let $A(z) = \sum_{n=0}^{\infty} t_n z^n$. Since $\sum_{n=2}^{\infty} t_n z^n - 5\sum_{n=2}^{\infty} t_{n-1} z^n + 6\sum_{n=2}^{\infty} t_{n-2} z^n = \sum_{n=2}^{\infty} 2^n z^n + \sum_{n=2}^{\infty} nz^n$,

we may write

$$A(z) - t_0 - t_1 z - 5z[A(z) - t_0] + 6z^2 A(z) = \frac{4z^2}{1 - 2z} + z\left[\frac{1}{(1 - z)^2} - 1\right]$$

Solving the above equation for $A(z)$, we get

$$A(z) = \frac{1 - 8z + 27z^2 - 35z^3 + 14z^4}{(1 - z)^2(1 - 2z)^2(1 - 3z)}$$

Breaking into partial fractions, we get

$$A(z) = \frac{5}{4(1 - z)} + \frac{1}{2(1 - z)^2} - \frac{3}{1 - 2z} - \frac{2}{(1 - 2z)^2} + \frac{17}{4(1 - 3z)}$$

The corresponding sequence is given by

$$t_n = \frac{5}{4} + \frac{1}{2}(n+1) - 3\times 2^n - 2(n+1)\times 2^n + \frac{17}{4}\times 3^n$$

$$= \frac{7}{4} + \frac{n}{2} - n\times 2^{n+1} - 5\times 2^n + \frac{17}{4}\times 3^n$$

since $1/(1-z)$ generates the sequence 1, 1, 1, ...

$1/(1-z)^2$ generates the sequence 1, 2, 3, ...

$1/(1-2z)$ generates the sequence 1, 2, 2^2, ...

$1/(1-2z)^2$ generates the sequence 1, 2×2, 3×2^2, ...

$1/(1-3z)$ generates the sequence 1, 3, 3^2, ...

EXAMPLE 1.20: Solve the following recurrence using the generating function

$$t_n = t_{n-1} + t_{n-2},\ n \ge 2, \text{ subject to } t_0 = 0,\ t_1 = 1.$$

Let
$$A(z) = \sum_{n=0}^{\infty} t_n z^n$$

Now,
$$t_n z^n = t_{n-1}z^n + t_{n-2}z^n$$

Therefore,
$$\sum_{n=2}^{\infty} t_n z^n = \sum_{n=2}^{\infty} t_{n-1}z^n + \sum_{n=2}^{\infty} t_{n-2}z^n$$

or
$$\sum_{n=2}^{\infty} t_n z^n = z\sum_{n=2}^{\infty} t_{n-1}z^{n-1} + z^2\sum_{n=2}^{\infty} t_{n-2}z^{n-2}$$

or
$$\sum_{n=2}^{\infty} t_n z^n = z\sum_{n=1}^{\infty} t_n z^n + z^2\sum_{n=0}^{\infty} t_n z^n$$

whereby,
$$A(z) - t_0 - t_1 z = z(A(z) - t_0) + z^2 A(z)$$

or
$$A(z) - z = zA(z) + z^2 A(z)$$

or
$$A(z)(1 - z - z^2) = z$$

or
$$A(z) = \frac{z}{1 - z - z^2}$$

$$= \frac{z}{\left(1 - \frac{1 - \sqrt{5}}{2} z\right)\left(1 - \frac{1 + \sqrt{5}}{2} z\right)}$$

$$= \frac{1}{\sqrt{5}} \left\{ \frac{1}{1 - \frac{1 + \sqrt{5}}{2} z} - \frac{1}{1 - \frac{1 - \sqrt{5}}{2} z} \right\}$$

The corresponding sequence is given by

$$t_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

since $1/(1 - cz)$ generates the sequence $1, c, c^2, \ldots$

1.7 A FAMOUS PROBLEM

We now look at the complexity of the Tower of Hanoi[4] problem. The problem is to move n discs from the first peg A (stacked in decreasing size) to the third peg C. The middle peg B may be used to hold discs during the transfer. The discs are to be moved one at a time under the condition that at any time, a disc of larger size cannot rest on a disc of smaller size on any of the pegs. For simplicity, let us consider that we have 3 discs on peg A initially as shown in Figure 1.1.

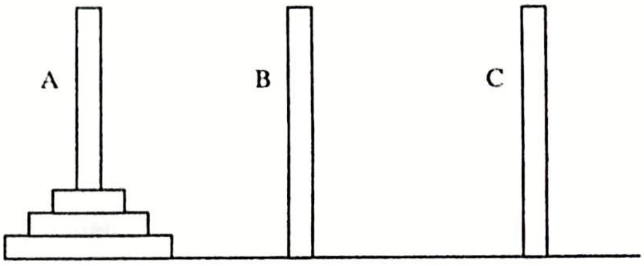


FIGURE 1.1 Tower of Hanoi.

The following sequence of moves (A $\rightarrow$ C means that the topmost disc from peg A is moved to peg C) will solve this instance of the Tower of Hanoi problem.

A $\rightarrow$ C, A $\rightarrow$ B, C $\rightarrow$ B, A $\rightarrow$ C, B $\rightarrow$ A, B $\rightarrow$ C, A $\rightarrow$ C

4. The legend says that the world will come to an end when 64 discs are moved from peg A to peg C. Refer Édouard Lucas, *Récréations Mathématiques*, Vol. 3, pp. 55–59, Albert Blanchard, Paris, 1960.

What about larger size problems? The problem admits a nice recursive solution. To move n discs from A to C, one can move $n - 1$ discs from A to B, the remaining disc from A to C and then move the $n - 1$ discs from B to C. The time complexity may be expressed in the form of the following recurrence relation:

$$T(n) = 2T(n - 1) + 1 \text{ with } T(1) = 1 \text{ and } T(2) = 3.$$

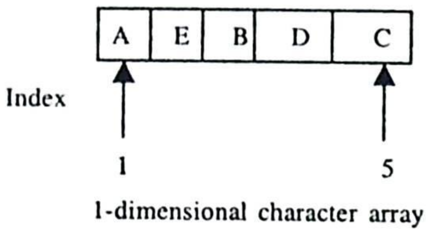
The solution of the above recurrence relation is $T(n) = A*2^n + B$. Putting the boundary conditions, we get $T(n) = 2^n - 1$. For $n = 64$, $2^n - 1 \approx 1.8 \times 10^{19}$, a large number indeed.

1.8 BASIC DATA STRUCTURES

In a programming language, the data type of a variable is the set of values that the variable may assume. For example, a variable of type character can assume values from the set. $\{x \mid x$ is an alphabetic character, any digit 0 through 9, and any of the permitted special symbols$\}$. A string type is constructed out of character type. An abstract data type (ADT) is a mathematical model, together with various operations defined on the model. To implement an algorithm in a given programming language, we have to find some way of representing the ADTs in terms of the data types and operators supported by the programming language. Data structures that are collection of variables of several data types aggregated in various ways, are used to represent the mathematical model underlying an ADT.

1.8.1 Array

The simplest data structure is an array. It is a collection of data storage cells of a particular type and may be accessed randomly through the use of a finite index set. Figure 1.2 shows a 1-dimensional character array and a 2-dimensional integer array.



$$\begin{bmatrix} 12 & 13 & 5 & 2 \\ 10 & 20 & 2 & 5 \\ 11 & 0 & 2 & 7 \\ 3 & 6 & 9 & 21 \end{bmatrix}$$

2-dimensional integer array

FIGURE 1.2 Array.

1.8.2 Linked List

The linked list consists of a series of cells (records) that are not necessarily adjacent in the memory. Each cell (record) contains the element and a pointer to a cell (record) containing its successor. A record is a group of fields. The common functions defined on a list include: deletion of a record, addition of a record, searching for a record, determining the length of the list, etc. There are many variations of linked lists, such as, two-way linked lists, circular lists, multi-lists, etc. Figure 1.3 shows one-way linked list.