Any sequence t_n which satisfies the recurrence is called a particular solution; any sequence which makes the left-hand side identically zero is called a homogeneous solution. Sometimes, we can guess a particular solution, but cannot easily find one which satisfies the boundary conditions.

Theorem 1.2: If we start with any particular solution t_n and add any homogeneous solution, we obtain another particular solution. Moreover, the difference between any two particular solutions is always a homogeneous solution.

The above theorem suggests the following approaches for solving inhomogeneous recurrences.

(1) Guess a particular solution t_n.
(2) Write a formula for t_n plus the general homogeneous solution with unknown constants.
(3) Use the boundary conditions to solve for the constants.

However, the above method has the disadvantage in the sense that we need to guess a particular solution. A fairly general technique for producing solutions to inhomogeneous equations with constant coefficients is to transform the inhomogeneous equation into a homogeneous equation through algebraic manipulations and then solve it employing the method used for solving homogeneous equations. This is illustrated through the next two examples.

EXAMPLE 1.13: Solve the following recurrence relation:

$$t_n = 2t_{n-1} + n$$

Rewriting, we have

$$t_n - 2t_{n-1} = n \tag{1.1}$$

Replacing n by $n + 1$, we get

$$t_{n+1} - 2t_n = n + 1$$

$$\Rightarrow \quad 2t_{n+1} - 4t_n = 2n + 2 \tag{1.2}$$

Replacing n by $n + 2$, we get

$$t_{n+2} - 2t_{n+1} = n + 2 \tag{1.3}$$

Adding Eqs. (1.1) and (1.3), we get

$$t_{n+2} - 2t_{n+1} + t_n - 2t_{n-1} = 2n + 2 \tag{1.4}$$

Subtracting Eq. (1.2) from Eq. (1.4), we get

$$t_{n+2} - 4t_{n+1} + 5t_n - 2t_{n-1} = 0$$

The characteristic equation is

$$x^3 - 4x^2 + 5x - 2 = 0$$

or

$$x^3 - 2x^2 - 2x^2 + 4x + x - 2 = 0$$

or $$x^2(x-2) - 2x(x-2) + 1(x-2) = 0$$

or $$(x-1)^2(x-2) = 0$$

The general solution is

$$t_n = c_1 2^n + c_2 1^n + c_3 n 1^n$$

whereby $$t_n = O(2^n)$$

EXAMPLE 1.14: Solve the following recurrence relation:

$$t_n = 2t_{n-1} + n + 2^n,\ n \geq 1, \text{ subject to } t_0 = 0.$$

Here, $$t_n - 2t_{n-1} = n + 2^n \tag{1.5}$$

Replacing n by $n + 1$, we get

$$t_{n+1} - 2t_n = n + 1 + 2^{n+1} \tag{1.6}$$

Replacing n by $n + 2$, we get

$$t_{n+2} - 2t_{n+1} = n + 2 + 2^{n+2} \tag{1.7}$$

Replacing n by $n + 3$, we get

$$t_{n+3} - 2t_{n+2} = n + 3 + 2^{n+3} \tag{1.8}$$

Multiplying Eq. (1.5) by –2, Eq. (1.6) by 5, Eq. (1.7) by –4 and Eq. (1.8) by 1 and then adding, we get

$$t_{n+3} - 6t_{n+2} + 13t_{n+1} - 12t_n + 4t_{n-1} = 0$$

The characteristic equation is

$$x^4 - 6x^3 + 13x^2 - 12x + 4 = 0$$

or $$x^4 - 4x^3 + 4x^2 - 2x^3 + 8x^2 - 8x + x^2 - 4x + 4 = 0$$

or $$(x^2 - 4x + 4)(x^2 - 2x + 1) = 0$$

or $$(x-2)^2(x-1)^2 = 0$$

This equation has roots 1 and 2, both of multiplicity 2. The general solution of the recurrence is of the form:

$$t_n = (c_1 + c_2 n)1^n + (c_3 + c_4 n)2^n$$

$\therefore$ $$t_n \in O(n2^n)$$

1.5 CHANGE OF VARIABLE

It is sometimes possible to solve more complicated recurrences by making a change of variable. The following three examples illustrate this.

EXAMPLE 1.15: Solve the recurrence: $T(n) = 4T(n/2) + n$, where $n \geq 1$, and is a power of 2.

Replacing n by 2^k and $T(2^k) = t_k$, we get

$$T(2^k) = 4T(2^{k-1}) + 2^k$$

or
$$t_k = 4t_{k-1} + 2^k \tag{1.9}$$

Replacing k by $k - 1$, we get

$$t_{k-1} = 4t_{k-2} + 2^{k-1} \tag{1.10}$$

Multiplying Eq. (1.10) by 2 and subtracting the result from Eq. (1.9), we get

$$t_k - 2t_{k-1} = 4t_{k-1} - 8t_{k-2} \quad \text{or} \quad t_k - 6t_{k-1} + 8t_{k-2} = 0$$

Putting $t_k = x^k$, we get

$$x^k - 6x^{k-1} + 8x^{k-2} = 0$$

or
$$x^2 - 6x + 8 = 0$$

or
$$(x - 2)(x - 4) = 0$$

$\therefore$
$$t_k = c_1 2^k + c_2 4^k$$

Putting back n, we get

$$T(n) = c_1 n + c_2 n^2$$

$\therefore$
$$T(n) \in O(n^2)$$

EXAMPLE 1.16: Solve the following recurrence relation

$$T(n) = 4T(n/2) + n^2$$

where $n > 1$ and is a power of 2.

Put $n = 2^k$ and $T(2^k) = t_k$ to get

$$t_k = 4t_{k-1} + 2^{2k} \tag{1.11}$$

and replacing k by $k - 1$, we get $t_{k-1} = 4t_{k-2} + 2^{2(k-1)}$ (1.12)

Multiplying Eq. (1.12) by 4 and subtracting the result from the Eq. (1.11), we get

$$t_k - 4t_{k-1} = 4t_{k-1} - 16t_{k-2}$$

or
$$t_k - 8t_{k-1} + 16t_{k-2} = 0$$

Putting $t_k = x^k$, we get

$$x^k - 8x^{k-1} + 16x^{k-2} = 0$$

Thus, the characteristic equation is

$$x^2 - 8x + 16 = 0 \quad \text{or} \quad (x - 4)^2 = 0$$

$$t_k = (c_1 + c_2 k)4^k$$

Putting back n, we get

$$T(n) = c_1n^2 + c_2n^2 \log n$$

$$\therefore \quad T(n) \in O(n^2 \log n)$$

EXAMPLE 1.17: Solve the recurrence: $T(n) = 7T(n/2) + 3n^2$, where n is a power of 2 and is greater than 1.

Let us put $n = 2^k$ and $T(2^k) = t_k$, to get

$$t_k = 7t_{k-1} + 3 \times 2^{2k} \tag{1.13}$$

Putting $k = k - 1$ in Eq. (1.13), we get

$$t_{k-1} = 7t_{k-2} + 3 \times 2^{2(k-1)} \tag{1.14}$$

Multiplying Eq. (1.14) by 4 and subtracting the result from Eq. (1.13), we get

$$t_k - 4t_{k-1} = 7t_{k-1} - 28t_{k-2}$$

or

$$t_k - 11t_{k-1} + 28t_{k-2} = 0$$

Putting $t_k = x^k$ in the above equation, we get $x^k - 11x^{k-1} + 28x^{k-2} = 0$. Thus, the characteristic equation is:

$$x^2 - 11x + 28 = 0$$

or

$$(x - 4)(x - 7) = 0$$

Hence,

$$t_k = c_14^k + c_27^k = c_1n^2 + c_27^{\log_2 n}$$

$$\therefore \quad T(n) = c_1n^2 + c_27^{\log_2 n}$$

1.6 GENERATING FUNCTIONS

Generating functions transform a problem from one conceptual domain to another, in the hope that the problem will be easier to solve in the new domain. We demonstrate how generating functions can be used to solve some simple recurrence relations.

EXAMPLE 1.18: Suppose we have a recurrence: $t_n = 2t_{n-1} + 1$, $n \ge 1$ and $t_0 = 1$.

Let

$$A(z) = \sum_{n=1}^{\infty} t_n z^n .$$

From the given recurrence this sum can be written as

$$A(z) = t_0 z^0 + \sum_{n=1}^{\infty} t_n z^n = 1 + \sum_{n=1}^{\infty} t_n z^n$$

$$= 1 + \sum_{n=1}^{\infty} (2t_{n-1} + 1) z^n$$