- $O(f(n)) + O(f(n)) = O(f(n))$
- $O(O(f(n))) = O(f(n))$
- $O(f(n)) * O(g(n)) = O(f(n) * g(n))$
- $O(f(n) * g(n)) = f(n) * O(g(n))$

EXAMPLE 1.9: Show that $n \log n \in \Theta(\log(n!))$.

We have, $\log(n!) = \log(1 \times 2 \times \ldots \times n)$

$$= \log(1) + \log(2) + \ldots + \log(n)$$
$$\leq \log(n) + \log(n) + \ldots + \log(n)$$
$$= n \log n$$

So, $$n \log n \in \Omega(\log(n!))$$

Similarly,

$$\log(n!) \geq \log(n/2) + \log(n/2 + 1) + \ldots + \log(n)$$

Therefore, $$\log(n!) \geq (n/2)\ \frac{1}{2} \log(n/2)$$

$$= (n/2)\ \frac{1}{2} \log(n) - (n/2)\ \frac{1}{2} \log(2)$$

That is, $\log(n!) \geq n \log(n)/4,\ n > 10$

So, $n \log(n) \in O(\log(n!))$

Hence, $n \log n \in \Theta(\log(n!))$

EXAMPLE 1.10: Show that if $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$.

From the definition, there exists N_1, N_2, K_1, K_2 such that

$$f(n) \leq K_1 g(n),\ n > N_1 \quad \text{and} \quad g(n) \leq K_2 h(n),\ n > N_2$$

Therefore, $$f(n) \leq K_1 K_2 h(n),\ n > max(N_1, N_2)$$

With $K_1 K_2 = K$ and $N_3 = max(N_1, N_2)$, we have

$$f(n) \leq Kh(n),\ n > N_3$$

Thus, $$f \in O(h)$$

1.2 SOLUTION OF SOME COMMON RECURRENCE RELATIONS

(i) Let the form of the recurrence be:

$$t_N = t_{N-1} + N,\ N \geq 2,\ t_1 = 1$$

$$t_N = t_{N-1} + N$$
$$= t_{N-2} + (N - 1) + N$$
$$= t_{N-3} + (N - 2) + (N - 1) + N$$

$= t_1 + 2 + 3 + \ldots + N$

$= 1 + 2 + 3 + \ldots + N$

$\therefore\ t_N = O(N^2)$

Let N be of the form 2^K for some positive integer K in the examples (ii), (iii) and (iv) below.

(ii) Let the form of the recurrence be:

$t_N = t_{N/2} + 1,\ N \geq 2,\ t_1 = 1$

$t_N = t_{N/2} + 1$

$= t_{N/4} + 1 + 1$

$\ldots$

$= t_1 + 1 + 1 + \ldots + 1$

$= \log N$

$\therefore\ t_N = O(\log N)$

(iii) Let the form of the recurrence be:

$t_N = t_{N/2} + N,\ N \geq 2,\ t_1 = 0.$

Then, $t_N = O(N)$.

The derivation is similar to (ii) above.

(iv) Let the form of the recurrence be:

$t_N = 2t_{N/2} + N,\ N \geq 2,\ t_1 = 0.$

Then $\frac{t_N}{N} = \frac{2t_{N/2}}{N} + 1$, whereby

$$\frac{t_{2^K}}{2^K} = \frac{2t_{2^{K-1}}}{2^K} + 1$$

$$= \frac{2t_{2^{K-1}}}{2^{K-1}} + 1$$

$$= \frac{t_{2^{K-2}}}{2^{K-2}} + 1 + 1$$

$\ldots$

$= 1 + 1 + 1 + 1 \ldots$ (to K terms)

$= K$

Then, $\frac{t_N}{N} = K \Rightarrow t_N = NK \Rightarrow t_N = N \log N \Rightarrow t_N = O(N \log N)$

1.3 HOMOGENEOUS RECURRENCES

A recurrence relation is said to be a homogeneous linear recurrence relation with constant coefficients if it has the form:

$$a_0t_n + a_1t_{n-1} + \dots + a_kt_{n-k} = 0$$

where a_i's are constants. An example is the recurrence: $t_n - t_{n-1} - 3t_{n-2} = 0$. We now discuss a general method for solving any such recurrence.

Let $t_n = x^n$, where x is a constant, as yet unknown. Then, we may write,

$$a_0x_n + a_1x^{n-1} + \dots + a_kx^{n-k} = 0$$

The equation is satisfied if $x = 0$ (a trivial solution of no interest) or else if, $a_0x^k + a_1x^{k-1} + \dots + a_k = 0$.

This is an equation of degree k in x and is called the *characteristic equation* of the recurrence.

Suppose for the time being that the k roots $r_1, r_2, \dots, r_k$ of this characteristic equation are all distinct (they could be complex numbers). It is then easy to verify that any linear combination of r_i's, is a solution of the characteristic equation and so of the recurrence, where the k constants $c_1, c_2, \dots, c_k$ are determined by the initial condition.

$$t_n = \sum_{i=1}^{k} c_ir_i^n$$

Theorem 1.1: Let $f(x)$ be the characteristic equation of the recurrence:

$$a_0t_n + a_1t_{n-1} + \dots + a_kt_{n-k} = 0$$

where the a_i are constants. Let the roots of f, over the complex numbers, be r_i, $i = 1, \dots, m$, and let their respective multiplicities be q_i, $i = 1, \dots, m$. Then any solution to the recurrence is of the form $\sum_{i=1}^{i=m}\left(r_i^n\sum_{j=0}^{q_i-1} c_{ij}n^j\right)$.

EXAMPLE 1.11: Consider the recurrence $t_n - 3t_{n-1} - 4t_{n-2} = 0$, $n \geq 2$, subject to $t_0 = 0$, $t_1 = 1$.

The characteristic equation is $x^2 - 3x - 4 = 0$ whose roots are -1 and 4. The general solution has the form:

$$t_n = c_1(-1)^n + c_24^n$$

The initial conditions give:

$$c_1 + c_2 = 0 \quad \text{and} \quad -c_1 + 4c_2 = 1$$

which gives,

$$c_1 = -1/5, \qquad c_2 = 1/5$$

$$\therefore \qquad t_n = \frac{4^n + (-1)^{n+1}}{5}$$

EXAMPLE 1.12: Fibonacci numbers.

$$t_n = t_{n-1} + t_{n-2},\ n \geq 2, \quad \text{subject to } t_0 = 0,\ t_1 = 1.$$

The recurrence may be written as:

$$t_n - t_{n-1} - t_{n-2} = 0$$

The characteristic equation is $x^2 - x - 1 = 0$. The roots[3] are: $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

The general solution is of the form:

$$t_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Initial conditions give:

$$0 = c_1 + c_2, \qquad 1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$$

The values of c_1 and c_2 are:

$$c_1 = \frac{1}{\sqrt{5}}, \quad c_2 = -\frac{1}{\sqrt{5}}$$

$$\therefore \qquad t_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

1.4 INHOMOGENEOUS RECURRENCES

A recurrence relation is said to be inhomogeneous, if it has the form:

$$a_0 t_n + a_1 t_{n-1} + \ldots + a_k t_{n-k} = F(n),$$

where $F(n)$ is a non-zero function of n.

As an example, the following is an inhomogeneous linear recurrence with constant coefficients:

$$t_n - 5t_{n-1} + 6t_{n-2} = 6$$

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[3] $(1+\sqrt{5})/2 \approx 1.61803$ is known as the golden ratio or the golden mean. This number is regarded as the most pleasing ratio in the art world since ancient times and is important in many parts of mathematics.