

**M.SC. MATHEMATICS
FIRST SEMESTER
REAL ANALYSIS
MSM – 101**
(USE OMR FOR OBJECTIVE PART)



Duration: 3 hrs.

Full Marks: 70

Time: 30 min.

(Objective)

Marks: 20

Choose the correct answer from the following:

1X20=20

- Let $S(x, r)$ be an open sphere in a discrete metric space (X, D) . Then $S(x, r)$ is a singleton set if
 - $0 < r < 1$
 - $0 < r \leq 1$
 - $r \geq 1$
 - $r > 1$
- Consider \mathbb{R} , the set of real numbers with usual metric d on \mathbb{R} given by $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. Then $S\left(-1, \frac{3}{2}\right)$ is equal to
 - $\left[-\frac{5}{2}, \frac{1}{2}\right]$
 - $\left[-\frac{5}{2}, \frac{1}{2}\right]$
 - $\left[-\frac{5}{2}, \frac{1}{2}\right]$
 - $\left[-\frac{5}{2}, \frac{1}{2}\right]$
- Let (X, d) be any metric space, and $A \subset X$. Then the interior of A is the
 - Intersection of all open sets contained in A .
 - Intersection of all open sets containing A .
 - Union of all open sets contained in A .
 - Union of all open sets containing A .
- Let (X, d) be any metric space, and $A \subset X$. Then the closure of A is the
 - Intersection of all closed sets contained in A .
 - Union of all closed sets contained A .
 - Intersection of all closed sets containing A .
 - Union of all closed sets containing in A .
- Let $\langle x_n \rangle$ be any sequence in a metric space (X, d) . If $\langle x_n \rangle$ converges then
 - the sequence is Cauchy
 - the sequence is not Cauchy
 - the sequence is not bounded
 - None of these is true
- Let $\sum f_n(x)$ be a series of continuous functions defined on $[a, b]$ for each n , converging pointwise to the sum function f . Then
 - f is continuous on $[a, b]$
 - f is discontinuous on some point in $[a, b]$
 - f may or may not be continuous on $[a, b]$
 - None of these

7. Let a sequence $\{f_n\}$ of real functions converges uniformly to a real function f so that given $\epsilon > 0$, there exists a positive integer m so that $|f_n(x) - f(x)| < \epsilon$, $\forall n \geq m$ for $x \in [a, b]$. Then
- m depends on $x \in [a, b]$ and not on ϵ
 - m depends on ϵ and not on $x \in [a, b]$
 - m is independent of both ϵ and $x \in [a, b]$
 - None of these
8. Let $\langle f_n \rangle$ be a sequence of functions such that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, $\forall x \in [a, b]$ and let $M_n = \sup_{x \in [a, b]} |f_n(x) - f(x)|$
- $M_n \rightarrow +\infty$ as $n \rightarrow \infty$
 - $M_n \rightarrow 0$ as $n \rightarrow \infty$
 - $M_n \rightarrow -\infty$ as $n \rightarrow \infty$
 - M_n is bounded for all n
9. The sequence $\langle f_n \rangle$ of functions where $f_n(x) = x^n$ defined on $[0, 1]$ is convergent to the limit function f where
- $f(x) = 1$, $\forall x \in [0, 1]$
 - $f(x) = 0$, $\forall x \in [0, 1]$
 - $f(x) = \begin{cases} 1, & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0 \end{cases}$
 - $f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$
10. Consider the series $\sum f_n$ of functions where $f_n(x) = \frac{x^2}{(1+x^2)^n}$, $x \in \mathbb{R}$. The series converges to a sum function f given by
- $f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$
 - $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$
 - $f(x) = 0$, $x \in \mathbb{R}$
 - $f(x) = 1$, $x \in \mathbb{R}$
11. For any interval $[a, b]$ in \mathbb{R} the length of $[a, b]$ is
- $a + b$
 - $a - b$
 - $b - a$
 - None of the above
12. If G is any open set in \mathbb{R} then
- G is union of a countable class of open intervals.
 - G is union of a disjoint class of open intervals.
 - G is union of a countable disjoint class of open intervals.
 - None of the above
13. For any set $A \subseteq [a, b]$, the outer measure m^*A is defined by
- $\sup l(F)$, where the supremum is taken over the length of all open sets $F \supseteq A$.
 - $\inf l(F)$, where the infimum is taken over the length of all open sets $F \supseteq A$.
 - $\inf l(F)$, where the infimum is taken over the length of all open sets $F \subseteq A$.
 - None of these
14. For any two subsets A_1 and A_2 in $[a, b]$
- $m^*A_1 + m^*A_2 \leq m^*(A_1 \cup A_2) + m^*(A_1 \cap A_2)$
 - $m^*A_1 + m^*A_2 \geq m^*(A_1 \cup A_2) + m^*(A_1 \cap A_2)$
 - $m^*A_1 + m^*A_2 \geq m^*(A_1 \cup A_2) + m^*(A_1 \cap A_2)$
 - None of these

15. If A be any subset of $[a, b]$ and m, A is the inner measure of A then given $\varepsilon > 0$, there is a closed set $G \subset A$ such that
- a. $m, A - \varepsilon < l(G)$
 - b. $m, A + \varepsilon < l(G)$
 - c. $m, A - \varepsilon > l(G)$
 - d. None of these
16. The radius of convergence of the power series $1 + 2x + 3x^2 + 4x^3 + \dots$ is
- a. 0
 - b. $\frac{1}{2}$
 - c. 1
 - d. 2
17. The power series $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ is
- a. Convergent at $x = 0$ only.
 - b. Everywhere Convergent
 - c. Nowhere Convergent
 - d. None of these
18. The radius of convergence R of a power series $\sum a_n x^n$ is given by
- a. $R = \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$
 - b. $R = \frac{1}{\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}}$
 - c. $R = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|^{\frac{1}{n}}$
 - d. $R = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$
19. The interval of convergence of the power series $1 + x^2 + x^4 + x^6 + \dots$ is
- a. $-1 \leq x < 1$
 - b. $-1 < x \leq 1$
 - c. $-1 < x < 1$
 - d. $-1 \leq x \leq 1$
20. The radius of convergence of the power series $x + \frac{x^2}{2^2} + \frac{2!}{3^3} x^3 + \frac{3!}{4^4} x^4 + \dots$
- a. $\frac{1}{e}$
 - b. e
 - c. $1 + e$
 - d. $1 - e$

(Descriptive)

Time : 2 hrs. 30 min.

Marks : 50

[Answer question no.1 & any four (4) from the rest]

1. a. Let (X, d) be any metric space. Define a metric d_1 on X by 5+2+1+
2=10
- $$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}, x, y \in X$$
- Show that (X, d_1) is again a metric space.
- b. Consider the sequence of the functions $\langle f_n \rangle$, where
- $$f_n(x) = \frac{\sin nx}{\sqrt{x}}, x \in \mathbb{R}$$
- Is $\langle f_n \rangle$ convergent? If so, find the limit function f for $\langle f_n \rangle$.
Examine the convergence of $\langle f'_n \rangle$, where $f'_n(x) = \frac{d}{dx} f_n(x)$,
 $x \in \mathbb{R}$.
2. a. When is a sequence $\langle x_n \rangle$ said to be convergent in a metric space (X, d) ? Prove that a convergent sequence in a metric space is always Cauchy. 1+3+1+
3+2=10
- Give an example to show that a Cauchy sequence in a metric space (X, d) may not be convergent.
- b. Consider the series of functions $\sum f_n$, where $f_n(x) = \frac{x^2}{(1+x^2)^n}$,
 $x \in \mathbb{R}$
- Examine the convergence of $\sum f_n$ and find the sum function f provided $\sum f_n$ is convergent. What is your observation on continuity of each term f_n and that of the sum function f ?
3. a. Prove Cauchy's criterion for uniform convergence of a series of functions $\sum f_n$ viz- 5+2+1+
2=10
- A series of functions $\sum f_n$ defined on an interval $I = [a, b]$ converges uniformly if and only if for $\varepsilon > 0$, and for all $x \in [a, b]$, there exists a positive integer m such that
- $$|f_{n+1}(x) + f_{n+2}(x) + \dots + f_{n+p}(x)| < \varepsilon, \forall n \geq m, p \geq 1$$

b. Examine the convergence of the sequence of function $\langle f_n \rangle$
 where $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$

Find the limit function f in case it is convergent. Also, in this case establish whether the convergence of the sequence is pointwise or uniform.

4. a. Prove Weierstrass's M-test viz - 5+5=10

A series of function $\sum f_n$ will converge uniformly and absolutely on $[a, b]$ if there is a convergent series $\sum M_n$ of positive numbers such that

for all $x \in [a, b]$, $|f_n(x)| \leq M_n \quad \forall n$.

b. Show that the series $\sum \frac{x}{n^p + x^2 n^q}$ converges uniformly over any finite interval $[a, b]$ for $0 < p \leq 1$, $p + q > 2$.

5. a. Let $\sum f_n$ be a sequence of functions converging uniformly to a 5+5=10
 limit function f in interval $[a, b]$. If f_n is continuous for each n in $[a, b]$, then prove that the limit f is also continuous in $[a, b]$.

b. Show that the series $\sum f_n$, where $f_n(x) = \frac{x^4}{(1+x^4)^{n-1}}$ is not uniformly continuous though it is pointwise convergent in $[0, 1]$.

6. a. Define radius of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$. 1+1+3+5=10
 Write a formula to find the radius of convergence R for $\sum a_n x^n$.

Hence find the radius of convergence for the power series
 $1 + 2x + 3x^2 + 4x^3 + \dots$

b. If a power series $\sum a_n x^n$ converges for $x = x_0$ then prove that it is absolutely convergent for every $x = x_1$ where $|x_1| < |x_0|$

7. a. Prove Abel's theorem on uniform convergence of a power 5+2+3=10
 series $\sum a_n x^n$ viz -

If a power series $\sum a_n x^n$ converges at end point $x = R$ of the interval $] -R, R[$ then it is uniformly convergent in the closed interval $[0, R]$.

b. Show that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots, -1 \leq x \leq 1$. Also show that $\frac{1}{2}(\tan^{-1} x)^2 = \frac{x^2}{2} - \left(1 + \frac{1}{3}\right)\frac{x^4}{4} + \left(1 + \frac{1}{3} + \frac{1}{5}\right)\frac{x^6}{6} - \dots, -1 \leq x \leq 1$

8. a. Define outer measure and inner measure of a set $A \subset [a, b]$. Hence show that $m_* A \leq m^* A$ 2+3+2+
2+1=10
- b. Prove that - If A_1 and A_2 are measurable sets in $[a, b]$ then both $A_1 \cup A_2$ and $A_1 \cap A_2$ are also measurable and
- $$m A_1 + m A_2 = m(A_1 \cup A_2) + m(A_1 \cap A_2)$$

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