REV-01
MSM/24/29

2023/06

SET A

M.Sc. MATHEMATICS
SECOND SEMESTER
TOPOLOGY & FUNCTIONAL ANALYSIS
MSM - 202

[USE OMR FOR OBJECTIVE PART]

Duration: 3 hrs. Full Marks: 70

(Objective)

Time: 30 min. Marks: 20

Choose the correct answer from the following: ***1X20=20***

1. Let $\mathbb{N}$ denotes the set of natural number and let $E_n = \{n, n+1, n+2, \cdots\}$ for $n \in \mathbb{N}$. If $\tau = \{\phi, \mathbb{N}, E_n\}$, then
   a. τ is not a topology on $\mathbb{N}$
   b. τ is a topology on $\mathbb{N}$
   c. is a topology on $\mathbb{N}$ for some $\in \mathbb{N}$
   d. None of these

2. Let $(X, \mathcal{T}_d)$ be a discrete topological space. Then which of the followings is/are true for $\mathcal{T}_d$?
   a. $\{\{x\}: \forall x \in X\}$ is a base
   b. $\{\{x\}: \forall x \in X\}$ is not a base
   c. $\{\{x\}: \forall x \in X\}$ is always a countable base
   d. $\{\{x\}: \forall x \in X\}$ is always a finite base

3. Let $Y = [0,1] \cup (2,3)$ be a subspace topology of $\mathbb{R}$. Then
   a. $[0,1]$ is closed in Y.
   b. $(2,3)$ is closed in Y.
   c. Both $[0,1]$ and $(2,3)$ are closed in Y.
   d. Neither $[0,1]$ nor $(2,3)$ is closed in Y.

4. Let $\mathcal{T}$ denote the usual topology on the real line $\mathbb{R}$. The subspace topology $\mathcal{T}_{\mathbb{N}}$ defined on the set of natural number is
   a. Equal to the discrete topology
   b. Equal to the indiscrete topology
   c. Neither discrete nor indiscrete
   d. $\mathcal{T}_{\mathbb{N}}$ is not a topology

5. Let $\mathcal{T}_1 = \{\phi, \{a\}, \{b,c\}, \{a,b,c\}\}$ be a topology on $X = \{a,b,c\}$ and $\mathcal{T}_2 = \{\phi, \{r\}, \{p,q\}, Y\}$ be a topology on $Y = \{p,q,r\}$. Find which of the following is/are closed mapping:
   a. $f(a) = r, f(b) = r, f(c) = r$
   b. $g(a) = p, g(b) = q, g(c) = p$
   c. $h(a) = r, h(b) = p, h(c) = q$
   d. All the above

6. Let X be a discrete topological space and $A \subset X$. Then ∂A is equal to
   a. empty set
   b. X
   c. proper subset of X
   d. Any subset of X

7. Which of the following is/are false
   a. Discrete topological space is 1st countable.
   b. The real line with lower limit topology is disconnected
   c. Every 1st countable space is 2nd countable.
   d. The real line with lower limit topology is 1st countable

8. Which of the following is/are open cover of the real line $\mathbb{R}^2$ with usual topology

a. $\mathcal{A} = \left\{B\left(a, \frac{1}{2}\right) : a \in \mathbb{Z} \times \mathbb{Z}\right\}$
b. $\mathcal{A} = \{B(a,1) : a \in \mathbb{Z} \times \mathbb{Z}\}$.
c. Both (a) and (b).
d. None of these

9. Consider the following statements:
P: The discrete topological space is a connected space.
Q: The indiscrete topological space is separable.

a. P true, Q false
b. P false, Q true
c. Both P and Q are true
d. None of these

10. Discrete topological space is

a. Compact.
b. Connected
c. Second Countable
d. None of these

11. Let $T: X \to Y$ be a linear operator from a normed space X into a normed space Y, both defined over the same field. Let us assume the statement
p: T is continuous at the origin.
Then p is equivalent to the statement

a. T is continuous on X
b. T is bounded on X
c. T maps bounded sets in X into bounded sets in Y
d. All the above

12. Let T be a bounded linear operator from a normed space X into a normed space Y, both X and Y being defined over the same field. Let S denote the statement

$$S : \| T \| = \sup\{\| T(x) \|_Y : \ x \in X, \| x \|_X \leq 1\}$$

Then S is equivalent to the statement:

a. $\| T \| = \sup\{\| T(x) \|_Y : x \in X, \| x \|_X = 1\}$
b. $\| T \| = \sup\left\{\frac{\| T(x) \|_Y}{\| x \|_X} : x \in X, x \neq 0\right\}$
c. $\| T \| = \inf\{K : K > 0 \text{ and } \| T(x) \|_Y \leq K \| x \|_X, \quad \forall x \in X\}$
d. All the above

13. A mapping $T: X \to Y$ from the normed space X into the normed space Y over the same field is isometrically isomorphic if

a. T is isometric
b. T is linear
c. T is linear and bijective
d. T is isometric, linear and bijective

14. Let X and Y be two normed spaces of dimensions m and n respectively over the same field K. X and Y are topologically isomorphic if

a. $m > n$
b. $m < n$
c. $m = n$
d. m is an integral multiple of n

15. Which of the following statement is true?

a. Two different norms on a finite dimensional normed space are equivalent.
b. Convergence or divergence of sequences in a finite dimensional normed space is dependent on the particular norm defined in it.
c. A finite dimensional subspace of a normed space is closed in .
d. An infinite dimensional subspace of a normed space is necessarily closed.

16. For any vector v in a normed vector space $(X, \|\cdot\|)$

a. $\| \quad \| \leq 0$
b. $\| v \| \geq 0$
c. $\| v \| < 0$
d. $\| v \| > 0$

17. For vector v in a normed vector space X and a scaler λ in the underlying field $\mathbb{R}$ of X

a. $\|\lambda x\| \leq |\lambda| \|x\|$
b. $\|\lambda x\| \geq |\lambda| \|x\|$
c. $\|\lambda x\| \neq |\lambda| \|x\|$
d. $\|\lambda x\| = |\lambda| \|x\|$

18. In a normed space $(X, \|\cdot\|)$ the closure of any subspace U of X is

a. Algebraically not a closed subspace.
b. Algebraically closed subspace.
c. Algebraically both open and closed.
d. Algebraically neither open nor closed.

19. Let $(X, \|\cdot\|)$ be a normed space over a field $\mathbb{R}$. For any fixed vector u and arbitrary vector x in X, define the translation function f_a such that $f_a(x) = a + x$. Then for any $x, y \in X$ (here d is the metric on X induced by the norm $\|\cdot\|$ in X)

a. $d((x), f_a(y)) \geq d(x, y)$
b. $d(f_a(x), f_a(y)) \leq d(x, y)$
c. $d(f_a(x), f_a(y)) = d(x, y)$
d. $d(f_a(x), f_a(y)) \neq d(x, y)$

20. For any two vectors x, y in a normed vector space $(X, \|\cdot\|)$

a. $| \| x \| - \| y \| | < \| x - y \|$
b. $| \| x \| - \| y \| | > \| x - y \|$
c. $| \| x \| - \| y \| | \leq \| x - y \|$
d. $| \| x \| - \| y \| | \geq \| x - y \|$

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Descriptive

Time : 2 hrs. 30 mins. **Marks : 50**

[Answer question no.1 & any four (4) from the rest]

1. a. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Define $\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}$. Show that $(A, \mathcal{T}_A)$ is a topological space. 4+2+4 =10

   b. Let A, B be the subspace topology of the topological space X and Y respectiviely. Prove that the product topology and subspace topology on $A \times B$ are same. Is the result true for order topology? Justify your answer.

2. a. Let X be any non-empty set. Define $\tau_f = \{A : X - A \text{ is finite or } X\}$ and τ_d be the discrete topology on X. 3+2+5 =10
   i) Show that τ_f is a topology on X.
   ii) Prove or disprove that $\tau_f \subset \tau_d$. Is $\tau_d \subset \tau_f$?

   b. Let (X, τ) be a topological space and let $\{A_\alpha : \alpha \in I\}$ be the collection of connected subspaces of X such that $\cap_{\alpha \in I} A_\alpha \neq \phi$. Prove that $\cup_{\alpha \in I} A_\alpha$ is connected.

3. a. Prove that – Every finite subspace of a topological space is compact. 3+5+2 =10

   b. Prove that - $(0, 1) \subseteq \mathbb{R}$ with usual topology is not compact.

   c. Prove or disprove that- Every discrete topological space is compact.

4. Verify the following statements: 5+5=10
   a. Every Discrete Topological Space is second countable.
   b. The lower limit topological space is second countable.

5. a. Let p be a real number with $1 \leq p < \infty$ and ℓ_p^n, the vector space of all n-tuples $f = (f(1), f(2), \cdots, f(n))$, $f(i) \in \mathbb{R}, i = 1, 2, \cdots, n$ in which norm of f is defined as 5+5=10

$$\| f \|_p = \left(\sum_{i=1}^{n} |f(i)|^p \right)^{1/p}$$

Verify if $(\ell_p^n, \|\cdot\|_p)$ is a normed space or not.

b. Consider ℓ_∞ the set of all bounded sequences in K, where K is $\mathbb{R}$ or $\mathbb{C}$.
For any $x = < x_i > \in \ell_\infty$, $x_i \in K$ define
$\| x \|_\infty = \sup\{|x_i|, x = < x_i >, x_i \in K\}$.
Prove that $(\ell_\infty, \|\cdot\|_\infty)$ is a Banach Space.

6. a. Let Y be any subspace of a normed space X. Show that if Y is complete then it is also closed in X. If Y is any closed subspace of a normed space X will it be complete in X ? Justify your answer. 3+3+4 =10

b. For any vectors x and y in a normed space $(X, \|\cdot\|)$, prove that

$\| \| x \| - \| y \| \| \leq \| x - y \|$.

7. a. Let X and Y be any normed spaces over a field K and let $B(X, Y)$ be the vector space of all bounded linear mappings from X into Y. Define $\|\cdot\|$ in $B(x, y)$ such that 5+1+4 =10
$\| T \| = \sup\{\| T(x) \|_Y : x \in X, \| x \|_X \leq 1\}$.
Show that $(B(X,Y), \|\cdot\|)$ is a normed space. Under what condition will $(B(X,Y), \|\cdot\|)$ will be a Banach space?

b. What do you mean by topological isomorphism for normed spaces $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$. Show that the two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ defined on the same linear space X will be equivalent if and only if there exist K_1 and K_2 satisfying

$$K_1 \| x \|_1 \leq \| x \|_2 \leq K_2 \| x \|_1, \quad \forall x \in X$$

8. a. Prove that any linear operator on a finite dimensional normed space is continuous. 5+5=10

b. State Hahn Banach Theorem and apply it to prove that if N is a normed linear space and x_0 is non-zero vector in N, then there is a function f_0 in N^* such that $f_0(x) = \| x_0 \|$ and $\| f_0 \| = 1$.

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