SET A

M.Sc. MATHEMATICS
FIRST SEMESTER
REAL ANALYSIS
MSM – 101 [REPEAT]

[USE OMR SHEET FOR OBJECTIVE PART]

Duration : 3 hrs. Full Marks : 70

(Objective)

Time: 30 min. Marks:20

Choose the correct answer from the following: ***1X20=20***

1. Which of the following is/ are locally connected?
   a. $\mathbb{Q}$is locally connected.
   b. $\mathbb{R}^n$is locally connected.
   c. Both $\mathbb{Q}$ and $\mathbb{R}^n$ are locally connected.
   d. None of these.

2. Which of the following function(s) is/are differentiable?
   a. $f(x) = |x|$
   b. $f(x) = \begin{cases} x \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$
   c. $f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$
   d. $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

3. Which of the following function(s) is/are true?
   a. The function $f(x) = x^2$ is uniformly continuous, in the interval $0 < x < \infty$
   b. The function $f(x) = \frac{1}{x}$ is uniformly continuous, in the interval $0 < x < 1$
   c. The function $f(x) = x^3$ is not uniformly continuous, in the interval $0 < x < 2$
   d. The function $f(x) = x^2$ is uniformly continuous, in the interval $0 < x < 1$

4. Let $X = \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$ be the unit circle inside $\mathbb{R}^2$. Let $f: X \to \mathbb{R}$ be a continuous function. Then
   a. Image f is compact
   b. Image f is compact
   c. The given information is not sufficient to determine whether Image f is compact.
   d. None of these.

5. Let $\{a_n\}, \{b_n\}$ and $\{C_n\}$ be sequence of real numbers such that $b_n = a_{2n}$and $C_n = a_{2n+1}$. Then $\{a_n\}$ is convergent
   a. If both $\{b_n\}$ and $\{C_n\}$ are convergent.
   b. Implies both $\{b_n\}$ and $\{C_n\}$ are convergent
   c. Implies $\{b_n\}$ is convergent but $\{C_n\}$ need not be convergent
   d. Implies $\{C_n\}$ is convergent but $\{b_n\}$ need not be convergent

6. Which of the following set is compact?
   a. $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$
   b. $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
   c. $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}$
   d. None of these

7. Which of the following is/ are true?

a. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=e$ and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=e+1$

b. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=e$ and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=e$

c. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=e$ but $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}$ does not exist.

d. Both $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}$ and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}$ does not exist.

8. Which of the following define a metric on $\mathbb{R}$.

(A) $d(x,y)=|x^2-y^2|$

(B) $d(x,y)=\frac{|x-y|}{1+|x-y|}$

(C) $d(x,y)=|x-2y|+|2y-x|$

a. Only (B) is true

b. Only (B) and (C) are true

c. All (A), (B) and (C) are true

d. Only (A) and (C) are true

9. Let f,g and h are bounded function on a closed interval $[a,b]$, such that $f(x)\le g(x)\le h(x), \forall x\in[a,b]$. Let $P=\{a=a_0<a_1<a_2<\cdots<a_n=b\}$ be a partition of $[a,b]$. $U(P,f)$ and $L(P,f)$ denotes the upper and Lower sums of f and similarly for g and h. Which of the following statements is necessarily true

a. If $U(P,h)-L(P,f)<1$ then $U(P,g)-L(P,g)<1$

b. If $L(P,h)-L(P,f)<1$ then $U(P,g)-L(P,g)<1$

c. If $U(P,h)-L(P,f)<1$ then $U(P,g)-L(P,g)<1$

d. If $L(P,h)-U(P,f)<1$ then $U(P,g)-L(P,g)<1$

10. If $f(x)=\frac{1}{1+e^{\frac{1}{x-2}}+e^{\frac{1}{(x-3)^2}}}$, if $x\neq 2$ and $x\neq 3$, $f(2)=1$ and $f(3)=\frac{1}{1+e}$, then

a. f is continuous at $x=2$ but not at $x=3$

b. f is continuous at $=3$ but not at $=2$

c. is neither continuous at $=2$ nor at $=3$

d. is continuous at $=2$ and $=3$ both

11. Which of the following is a correct statement:

a. Inverse image of a compact metric space under a continuous function is compact

b. Image of a compact metric space is compact.

c. Every closed and bounded subset of a compact metric space is compact.

d. The closed set $[0,\infty]$ is compact.

12. If $\{x_n\}$ is a convergent sequence in $\mathbb{R}$ and $\{y_n\}$ is a bounded sequence in $\mathbb{R}$, then

a. $\{x_n+y_n\}$ is convergent

b. $\{x_n+y_n\}$ is bounded

c. $\{x_n+y_n\}$ has no convergent subsequence

d. $\{x_n+y_n\}$ has no bounded subsequence

13. Let $\{a_n\}=\{n^2\}$ and $\{b_n\}=\{-n^2\}$ be two real sequence, then

a. The sequence $\left\{\frac{a_n}{b_n}\right\}$ is convergent but the sequence $\{a_n\}$ and $\{b_n\}$ are divergent.

b. The sequence $\left\{\frac{a_n}{b_n}\right\}$ is divergent but the sequence $\{a_n\}$ and $\{b_n\}$ are convergent.

c. The sequence $\left\{\frac{a_n}{b_n}\right\}$ and $\{a_n\}$ are convergent but the sequence $\{b_n\}$is divergent.

d. None of the sequences $\left\{\frac{a_n}{b_n}\right\}$, $\{a_n\}$ and $\{b_n\}$ is convergent.

14. Which one of the following statements is true?

a. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is divergent

b. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent

c. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is convergent

d. $\sum_{n=1}^{\infty} n^3$ is divergent

15. Which of the following is/ are correct?

a. $(0,1)$is connected but not compact.

b. $(0,1)$is compact but not connected.

c. $(0,1)$is both connected and compact.

d. None of these

16. If $u = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$ and $v = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$ then

a. $u = v = \infty$

b. $u = \infty, v = 0$

c. $u = 0, v = \infty$

d. $u = 0, v = 0$

17. Which of the following is/ are correct?

a. $f(x) = x\sin\frac{1}{x}$ is continuous but not differentiable on $(0,1)$

b. $f(x) = x\sin\frac{1}{x}$is differentiable but not continuous on $(0,1)$

c. $f(x) = x\sin\frac{1}{x}$ is both continuous and differentiable on $(0,1)$

d. None of these

18. Consider the following statement:

P: Every continuous function on $[a,b]$ is Riemann integrable.

Q: $f(x) = \begin{cases} 1, & \text{if } x \in \left[0,\frac{1}{2}\right] \\ 0, & \text{if } x \in \left(\frac{1}{2},1\right] \end{cases}$ is Riemann Integrable.

a. P is true but Q is false

b. Both P and Q are true, but P is not the correct explanation of Q

c. Both P and Q are true and P is the correct explanation of Q

d. Q is true but P is false

19. Consider the following two statements

P : Every Contraction is Lipschtiz continuous function.

Q : Every Lipschitz continuous function is Contraction.

a. P false, Q true

b. P true. Q false.

c. Both P & Q are true.

d. Both P & Q are false

20. The derive set of $(2,5) \cup \{7\}$ is

a. $(2,5) \cup \{7\}$

b. $[2,5] \cup \{7\}$

c. $(2,5)$

d. $[2,5]$

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Descriptive

Time : 2 hrs. 30 mins. Marks : 50

[Answer question no.1 & any four (4) from the rest]

1. a. Show that x^2 is integrableon [0,1]. 4+6=10
   b. Show that the metric space $\mathbb{R}$ with usual metric is connected.

2. a. Let$f(x) = x^2$. For each$n \in \mathbb{N}$, let σ_n be the partition $\left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}\right\}$ of $[0,1]$. Compute $\lim_{n\to\infty} U(\sigma_n; f)$ and$\lim_{n\to\infty} L(\sigma_n; f)$. 5+2+3 =10

   b. With the usual metrics on [0,1] and [0,2], find whether the function $f: [0,1] \to [0,2]$ define by $f(x) = ax + b$ is isometry.

   c. Let $f: X \to X$ be a function defined as $f(x) = x^2$ where $X = \left[0, \frac{1}{3}\right]$ is a metric space with usual metric. Prove or disprove that f is contraction. Is f is Lipschitz continuous function?

3. a. Which of the following define a metric on $\mathbb{R}$? 4+4+2 =10

$$(a)\ d_1(x,y) = |x^2 - y^2|$$
$$(b) d_2(x,y) = |x^3 - y^3|$$
$$(c) d_3(x,y) = d_1(x,y) + d_2(x,y)$$

   b. Find all the limit points of $(0,1)$.

   c. Let $I_n = \left\{\left(\frac{-1}{n}, \frac{1}{n}\right) : n \in \mathbb{R}\right\}$ and $(\mathbb{R}, d)$ be usual metric space. Prove or disprove that $\cap\, I_n$ is open.

4. a. Show that the series $\sum \frac{3\cdot6\cdot9\cdots3n}{7\cdot10\cdot13\cdots(3n+4)} x^n$, $x > 0$ converges for $x \le 1$, and diverges for $x > 1$. 5+5=10

   b. Examine the convergence of $\sum \frac{1}{2^n}, n \ge 1$.

5. a. Evaluate 5×2=10

   (i) $\lim_{n\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

   (ii) $\lim_{n\to 0} \left(1 - \frac{1}{x}\right)^x$

(iii) $\lim_{n\to 0}\frac{\sin x}{e^x-1}$
(iv) $\lim_{n\to 0}\frac{\sec^2 x-2\tan x}{1+\cos x}$
(v) $\lim_{n\to 0}\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$

6. a. Prove that for any real number k, the polynomial given by $f(x) = x^3 + x + k$ has exactly one real root. 4+4+2 =10

b. If $f(x) = \begin{cases} x^3 \text{for } x < 1 \\ ax + b \text{ for } x > 1 \end{cases}$, find the values of a and b for which $f(x)$ is differentiable at $x = 1$.

c. Prove that the discrete metric space $(X, d)$is compact if X is finite and it is not compact if X is infinite

7. a. Find the closure and derived set of $\mathbb{Q}$. 2+3+5 =10

b. Prove that if $\frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + \frac{a_{n-1}}{2} + a_n = 0$, then the equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ has at least one root between 0 and 1.

c. Prove that the following functions are uniformly continuous:
(i) $f(x) = \frac{x}{x+2}$ in $[0, 2]$
(ii) $f(x) = x^2 + 4x$ in $[-1, 1]$

8. a. If $s_1 > s_2 > 0$ and if $s_{n+1} = \frac{1}{2}(s_n + s_{n-1})$ for $n \geq 2$. Prove that : 6+4=10
(i) $s_1, s_3, s_5, \cdots$is monotonic decreasing.
(ii) $s_2, s_4, s_6, \cdots$is monotonic increasing.
(iii) (s_n) is convergent and$\lim_{n\to\infty}\frac{1}{3}(s_1 + 2s_2)$.

b. Let $s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$. Prove that $\lim_{n\to\infty} s_n$ exists and lies between 2 and 3.

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