REV-01
MSM/28/14/19

Gc

2023/01

SET A

M.Sc. MATHEMATICS
FIRST SEMESTER
REAL ANALYSIS
MSM – 101

[USE OMR SHEET FOR OBJECTIVE PART]

Duration : 3 hrs. **Full Marks : 70**

(Objective)

Time: 30 min. **Marks: 20**

Choose the correct answer from the following: ***1X20=20***

1. Let $S(x, r)$ be an open sphere in a discrete metric space $(X, \mathcal{D})$. Then $S(x, r)$ is a singleton set if
   - a. $0 < r < 1$
   - b. $0 < r \leq 1$
   - c. $r \geq 1$
   - d. $r > 1$

2. Consider $\mathbb{R}$, the set of real numbers with usual metric d on $\mathbb{R}$ given by $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. Then $S\left(1, \frac{1}{2}\right)$ is equal to
   - a. $\left(\frac{1}{2}, \frac{3}{2}\right]$
   - b. $\left[\frac{1}{2}, \frac{3}{2}\right)$
   - c. $\left(\frac{1}{2}, \frac{3}{2}\right)$
   - d. $\left[\frac{1}{2}, \frac{3}{2}\right]$

3. Let A' be the set of all limit points of $A \subset X$, where (X, d) is a metric space. Then A is said to be a closed set if
   - a. $\supset '$
   - b. $\not\supset '$
   - c. $A \neq A'$
   - d. None of these

4. Let (X, d) be any metric space, and $A \subset X$. Then the interior of A is the
   - a. Intersection of all open sets contained in A.
   - b. Intersection of all open sets containing A.
   - c. Union of all open sets containing A.
   - d. Union of all open sets contained in A.

5. Let $\{x_n\}$ be any sequence in a metric space (X, d). If $\{x_n\}$ converges then
   - a. It is a Cauchy sequence
   - b. It is not a Cauchy sequence
   - c. It has a subsequence which is not convergent
   - d. None of these

6. In a metric space (X, d)
   - a. ϕ is an open but X is not open
   - b. X is an open but ϕ is not open
   - c. Neither ϕ nor X is open
   - d. Both ϕ and X are open

7. Let $\{f_n(x)\}$ be a sequence of functions defined on an interval $[a, b]$ converging pointwise to the limit function f and $f_n(x)$ is bounded for each n in $[a, b]$. Then
   - a. f is bounded on $[a, b]$
   - b. f is unbounded on $[a, b]$
   - c. f may or may not be bounded on $[a, b]$
   - d. None of these

8. Let $\sum f_n(x)$ be a series of continuous functions defined on $[a,b]$ for each n, converging pointwise to the sum function f. Then

a. f is continuous on $[a, b]$

b. f is discontinuous on $[a,b]$

c. f may or may not be continuous on $[a, b]$

d. None of these

9. Let a sequence $\{fn\}$ of real functions converges uniformly to a real function f so that given $\epsilon>0$, there exists a positive integer m so that $fnx-fx<\epsilon$, $\forall n\geq m$ for $x\in[a, b]$. Then

a. m depends on $x\in[a, b]$ and not on ϵ

b. m depends on ϵ and not on any $x \in [a,b]$

c. m is independent of both ϵ and $x\in[a, b]$

d. None of these

10. The sequence $\{f_n(x)\}$ of functions where $f_n(x) = x^n$ defined on $[0,1]$ is convergent to the limit function f where

a. $f(x) = 1, \quad \forall x \in [0,1]$

b. $f(x) = 0, \quad \forall x \in [0,1]$

c. $f(x) = \begin{cases} 1, & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0 \end{cases}$

d. $f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

11. Let $\{f_n\}$ be a sequence of functions such that $\lim_{n\to\infty} f_n(x) = f(x), \ \forall x \in [a,b]$ and let $M_n = \text{Sup}_{x\in[a,b]}|f_n(x) - f(x)|$. Then $f_n \to f$ uniformly on $[a,b]$ if and only if

a. $M_n \to +\infty$ on $n \to \infty$

b. $M_n$issimply convergent

c. $M_n \to -\infty$ on $n \to \infty$

d. M_n is bounded for all n

12. Consider the series $\sum f_n$ of functions where $f_n(x) = \frac{x^2}{(1+x^2)^n}$, $x \in \mathbb{R}$. The series converges to a sum function f given by

a. $f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$

b. $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$

c. $f(x) = 0, \quad x \in \mathbb{R}$

d. $f(x) = 1, \quad x \in \mathbb{R}$

13. If F is any open set in $\mathbb{R}$ with usual metric then

a. F is union of a countable class of open intervals.

b. F is union of a disjoint class of open intervals

c. F is union of a countable disjoint class of open intervals

d. None of the above

14. If F is any open set in $[a,b]$ then $l(F) = \sum_{n=1}^{\infty} l(I_n)$ where

a. $F = \bigcup_{n=1}^{\infty} I_n, \quad I_n \subseteq [a,b]$

b. $F = \bigcup_{n=1}^{\infty} I_n$, $I_n \subseteq [a,b]$ and $I_i \cap I_j = \phi$ for $i \neq j$.

c. $F = \bigcup_{n=1}^{\infty} I_n$, $I_n \subseteq [a,b]$ and $I_i \cup I_j = [a,b]$ for $i \neq j$.

d. None of the above

15. For any set $A \subseteq [a,b]$, the outer measure m^*A is defined by

a. Sup $l(F)$ where the supremum is taken over the length of all open supersets of A.

b. Inf $l(F)$ where the infimum is taken over the length of all open supersets of A.

c. Sup $l(F)$ where the supremum is taken over the length of all open subsets of A

d. None of these

16. If A be any subset of $[a, b]$ and $m_* A$ is the inner measure of A then given $\mathcal{E} > 0$, there is a closed set $G \subset A$ such that

a. $m_* A - \mathcal{E} < l(G)$

b. $m_* A + \mathcal{E} < l(G)$

c. $m_* A - \mathcal{E} > l(G)$

d. None if these

17. For any two subsets A_1 and A_2 in $[a, b]$,

a. $m^* A_1 + m^* A_2 \leq m^*(A_1 \cup A_2) + m^*(A_1 \cap A_2)$

b. $m^* A_1 + m^* A_2 \geq m^*(A_1 \cup A_2) + m^*(A_1 \cap A_2)$

c. $m_* A_1 + m_* A_2 \geq m_*(A_1 \cup A_2) + m_*(A_1 \cap A_2)$

d. None of these

18. The power series $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ is

a. Convergent for all positive real numbers

b. Convergent for all negative real numbers

c. Convergent everywhere in $\mathbb{R}$

d. Convergent nowhere in $\mathbb{R}$ except $x = 0$

19. The power series $1 + x + 2!\,x^2 + 3!\,x^3 + 4!\,x^4 + \cdots$ is

a. Convergent for all real $x > 0$.

b. Convergent for all real $x < 0$.

c. Convergent for all real $x \in \mathbb{R}$.

d. Convergent for nowhere in $\mathbb{R}$ except $x = 0$.

20. If $\overline{\lim}|a_n|^{\frac{1}{n}} = \frac{1}{R}$ then the series $\sum a_n x^n$ is convergent for

a. $|x| > R$

b. $|x| \geq R$

c. $|x| < R$

d. None of these

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Descriptive

Marks : 50

Time : 2 hrs. 30 mins.

[Answer question no.1 & any four (4) from the rest]

1. a. Let (X,d) be a metric space and $A \subseteq X$. What do you mean by limit point and isolated point of the set A. 2+2+3+3=10

   b. Consider $\mathbb{R}$, the set of real numbers with usual metric d given by $d(x,y) = |x-y|$. Find the limit points and isolated points of $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ in $(\mathbb{R}, d)$.

   c. Define $\bar{A}$, the closure of a set A in a metric space (X,d) and prove that $\bar{A}$ is the smallest closed set containing A in (X,d).

   d. Show that a convergent sequence in a metric space is a Cauchy sequence and cite an example to show that a Cauchy sequence may not be convergent in a metric space.

2. a. Prove that any subsequence of a convergent sequence in a metric space is convergent. 5+5=10

   b. Explain the concept of pointwise convergence of a sequence of functions $\{f_n\}$ of real numbers with illustration.

3. a. Define the uniform convergence of a sequence $\{f_n\}$ and a series $\sum f_n$ of functions of real numbers on an interval $[a,b]$. (2+2)+(3+3)=10

   b. Prove that the necessary and sufficient condition for a sequence of function $\{f_n\}$ to converge uniformly on an interval $I = [a,b]$ is that for every $\epsilon > 0$ and for all $x \in I$, there is an integer m such that

   $$|f_{n+p}(x) - f_n(x)| < \epsilon, \forall n \geq m, p \geq 1$$

4. a. Prove that a series of function $\sum f_n$ will converge uniformly and absolutely on $[a,b]$ if there is a convergent series $\sum M_n$ of positive numbers such that for all $x \in [a,b]$, $|f_n(x)| \leq M_n$, $\forall n$. 6+4=10

   b. Apply part (a) above to show that $\sum \frac{\sin(x^2+n^2x)}{n(n+1)}$ is uniformly convergent for all $x \in \mathbb{R}$.

5. a. If a series $\sum f_n$ of functions converges uniformly to a function f in an interval $[a,b]$ and its terms f_n are all continuous at a point $x_0 \in [a,b]$ then prove that the sum function f is also continuous at x_0. 6+4=10

b. Apply part (a) to show that the series of functions:

$$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \frac{x^4}{(1+x^4)^3} + \cdots$$

is not uniformly continuous on $[0, 1]$.

6. a. What do you mean by the radius of convergence of power series $\sum_{x=0}^{\infty} a_n x^n$? Mention the formula for the radius of convergence of the power series $\sum_{x=0}^{\infty} a_n x^n$. **2+2+3+3=10**

b. Find the radii of convergence of the following series:

(i) $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(ii) $\frac{1}{2}x + \frac{1\cdot3}{2\cdot5}x^2 + \frac{1\cdot3\cdot5}{2\cdot5\cdot8}x^3 + \cdots$

7. a. Prove Abel's theorem viz, "If a power series $\sum a_n x^n$ converges at the end point $x = R$ of the interval of convergence $(-R, R)$ then it uniformly convergent is the closed interval $[0, R]$." **5+2+3=10**

b. Define outer and inner measure of a set $A \subset [a, b]$. Hence show that $m_* A \leq m^* A$.

8. a. If A_1 and A_2 are measurable sets in $[a, b]$ then prove that both $A_1 \cup A_2$ and $A_1 \cap A_2$ are also measurable and $mA_1 + mA_2 = m(A_1 \cup A_2) + m(A_1 \cap A_2)$. **6+4=10**

b. If A_1 and A_2 are measurable subsets of $[a, b]$ then prove that $A_1 - A_2$ is also measurable. Also, prove that if $A_2 \subset A_1$ then $m(A_1 - A_2) = mA_1 - mA_2$.

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