2023/01

SET A

M.Sc. MATHEMATICS
FIRST SEMESTER
LINEAR ALGEBRA
MSM – 102

[USE OMR SHEET FOR OBJECTIVE PART]

Duration : 3 hrs. Full Marks : 70

(Objective)

Time: 30 min. Marks: 20

Choose the correct answer from the following: ***1X20=20***

1. Consider the field $\mathbb{R}$ of all real numbers. Then $\mathbb{R}$ is a vector space over
   a. $\mathbb{Q}$
   b. $\mathbb{Z}$
   c. $\mathbb{N}$
   d. None of these

2. Which of the following sets is/are linearly independent in $\mathbb{R}^3$:
   a. $\{(1,2,5),(1,3,1),(2,5,7),(3,1,4)\}$
   b. $\{(1,2,5),(2,5,1),(1,5,2)\}$
   c. $\{(1,2,3),(0,0,0),(1,5,6)\}$
   d. None of these

3. Let W_1 and W_2 be two subspaces of a vector space V over a field F. Then
   a. $W_1 + W_2$ is not a subspace of V.
   b. $W_1 \cup W_2$ is a subspace of V.
   c. $W_1 \cap W_2$ is a subspace of V.
   d. $W_1 \cap W_2$ is not a subspace of V.

4. Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator defined by $F(x,y) = (2x+3y, 4x-5y)$. The matrix representation of F relative to the usual basis $\{(1,0),(0,1)\}$ is
   a. $\begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}$
   b. $\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$
   c. $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$
   d. $\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$

5. The determinant of the matrix $\begin{bmatrix} 2 & 3 & 4 & 7 & 8 \\ -1 & 5 & 3 & 2 & 1 \\ 0 & 0 & 2 & 1 & 5 \\ 0 & 0 & 3 & -1 & 4 \\ 0 & 0 & 5 & 2 & 6 \end{bmatrix}$
   a. 370
   b. -377
   c. 377
   d. -370

6. Let $S = \{u_1, u_2, \cdots, u_n\}$ be a set of linearly independent vectors of a vector space V of dimension n over a field F. Then
   a. Every superset of S is linearly independent.
   b. Every subset of S is linearly independent.
   c. No subset of S is linearly independent.
   d. Any subset of S may be linearly dependent.

7. A non-empty subset B of a vector space V over a field F is a basis of V if
   a. B generates V and is linearly dependent
   b. B is linearly independent and is not a generating set for V
   c. B is linearly dependent and is a generating set for V
   d. B generates V and is linearly independent

8. The matrix A whose minimal polynomial is $f(t) = t^3 - 8t^2 + 5t + 7$ is

a. $\begin{bmatrix} 0 & 0 & 7 \\ 1 & 0 & 5 \\ 0 & 1 & -8 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 0 & 7 \\ 0 & 1 & 5 \\ 0 & 1 & -8 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 0 & -7 \\ 1 & 0 & -5 \\ 0 & 1 & 8 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 0 & -7 \\ 1 & 0 & -5 \\ 0 & 1 & 8 \end{bmatrix}$

9. Let B and B' be two bases of the same vector space V having m and n elements respectively. Then

a. $m > n$ b. $m < n$

c. $m = n$ d. None of these

10. Consider $\mathbb{R}^2 = \{(x, y): x, y \in \mathbb{R}\}$ the vector space of all ordered pairs of real numbers. Then the base for $\mathbb{R}^2$ is given by

a. $\{(0,0), (1,0)\}$ b. $\{(0,0), (0,1)\}$

c. $\{(0,0), (1,1)\}$ d. $\{(1,0), (0,1)\}$

11. Let $(X, <,>)$ be an inner product space over the real field. If $u, v \in X$ and $< u, v > = 0$. Then

a. $u = 0, v \neq 0$ b. $u \neq 0, v = 0$

c. $u = 0, v = 0$ d. Both u and v may be nonzero

12. Define inner product $< u, v >$ on $\mathbb{R}^2$ as

$$< , >= {}_{11} - {}_{12} - {}_{21} + 3_{22}$$

Where $= ({}_1, {}_2)$ and $= ({}_1, {}_2)$. For $= (-1,2)$, $= (2,1)$, $= (1,-1)$

a. $< , > = 0$ b. $< , > = 10$

c. $< , > = 0$ d. $< , > = -3$

13. A real symmetric matrix is positive definite if

a. $< , >> 0$ for all nonzero vector $\in \mathbb{R}$ b. $< , >> 0$ for all nonzero vector $\in \mathbb{R}$

c. $< , >> 0$ for all nonzero vector $\in \mathbb{R}$ d. None of these

14. Projection of $= (1,3,5,7)$ upon $= (1,1,1,1)$ is

a. 2 b. 3

c. 4 d. 5

15. For any subset of a real inner product space V

a. $S \cap S^\perp = \phi$ b. $S \cap S^\perp = V$

c. $S \cap S^\perp \neq \{0\}$, where 0 is the zero vector of V. d. $S \cap S^\perp = \{0\}$, where 0 is the zero vector of V.

16. The real symmetric matrix A is positive definite if

a. $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ b. $A = \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$ d. $A = \begin{bmatrix} 6 & 7 \\ 7 & 9 \end{bmatrix}$

17. Let λ be an eigenvalue of a linear operator T on an inner product space V. Let $T^* = -T$. Then

a. λ is real b. λ is purely imaginary

c. $|\lambda| = 1$ d. None of these

18. The eigenvalue of the matrix $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 4 & 0 & 1 \\ 3 & 1 & 5 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are

a. 3,2,3,4
b. 4,5,1,0
c. 0,3,4,5
d. 1,2,4,5

19. Let A be 3×3 matrix with eigenvalues $1, -1$ and 3. Then

a. $A^2 + A$ is non-singular
b. $A^2 - A$ is non-singular
c. $A^2 + 3A$ is non-singular
d. $A^2 - 3A$ is non-singular

20. The minimal polynomial of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is

a. $(x-1)^2(x-2)$
b. $(x-1)(x-2)^2$
c. $(x-1)(x-2)$
d. $(x-1)^2(x-2)^2$

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Descriptive

Time : 2 hrs. 30 mins. Marks : 50

[Answer question no.1 & any four (4) from the rest]

1. a. Define an Inner product space V both over the field of real as well as complex numbers. 2+2+4+2=10
   b. Prove Schwarz's inequality
   $$|< x,y >| \leq \| x \| \| y \|$$
   For an inner product space $(V, <,>)$ over the field $\mathbb{R}$ of real numbers.
   c. Examine if the three vectors in $\mathbb{R}^3$ given by $u_1 = (1,2,1), u_2 = (2,1,-4)$ and $u_3 = (3,-2,1)$ are orthogonal or not.

2. Let $B = \begin{bmatrix} 11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$ 2+4+2=10
   (a) Find all eigenvalues of B.
   (b) Find a maximal set of non-zero orthogonal eigenvectors of B.
   (c) Find an orthogonal matrix P such that $D = P^{-1}BP$ is diagonal.

3. Using variables s and t, find an orthogonal substitution that diagonalizes the following quadratic forms: 10
   $$q(x,y) = 4x^2 + 8xy - 11y^2$$

4. Find the characteristic polynomial and minimal polynomial of each of the matrix: 4+2+4=10

   (a) $A = \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}$

   (b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

   (c) $C = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

5. a. Determine whether or not each of the following form a basis 6+4=10
   (i) $\{(1,1,1), (1,2,3), (2,-1,1)\}$ over $\mathbb{R}^3$
   (ii) $\{(1,1,2), (1,2,5), (5,3,4)\}$ over $\mathbb{R}^3$
   (iii) $\{(1,1,1,1), (1,2,3,2), (2,5,6,4), (2,6,8,5)\}$ over $\mathbb{R}^4$

b. Determine whether or not W is a subspace of $\mathbb{R}^3$, where
(i) $W = \{(a,b,c) \in \mathbb{R}^3 : a + b + c = 0\}$
(ii) $W = \{(a,b,c) \in \mathbb{R}^3 : a = 2b = 3c\}$

6. a. Apply the Gram-Schmidt orthogonalization process to find an orthogonal basis and then an orthonormal basis for the subspace U of $\mathbb{R}^4$ spanned by **4+1+3+2=10**

$$u_1 = (1,1,1,1), u_2 = (1,2,4,5), u_3 = (1,-3,-4,-2).$$

b. Define positive definite matrix A. Examine if the following matrices are positive definite or not:

$A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 \\ -2 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

(d) Find the angle between the vectors $u = (2,3,5)$, $v = (1,-4,3)$ in $\mathbb{R}^3$.

7. a. For a linear operator T on an inner product space V, define its adjoint T^* on V. **2+4+4=10**

b. If T_1 and T_2 are linear operators on an inner product space V over a field $\mathbb{R}$ then prove that
(i) $(T_1 + T_2)^* = T_1^* + T_2^*$
(ii) $(T_1T_2)^* = T_2^*T_1^*$

c. Let λ be an eigenvalue of a linear operator T on an inner product space V over the field $\mathbb{R}$. Prove that
(i) if $T^* = T^{-1}$ then $|\lambda| = 1$.
(ii) if $T^* = T$ then λ is real.

8. a. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator defined by **5+5=10**

$$T(x,y) = (2x + 3y, 4x - 5y)$$

Then find the matrix representation of T relative to the basis $\beta = \{u_1, u_2\}$, where $u_1 = (1,2), u_2 = (2,5)$.

b. Consider the two bases of $\mathbb{R}^2$ viz
$\mathcal{B} = \{(1,2),(3,5)\}$ and $\mathcal{B}' = \{(1,-1),(1,-2)\}$. Find the change-of-basis matrix P from $\mathcal{B}$ to $\mathcal{B}'$.

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