CHAPTER 1

INTRODUCTION AND MOTIVATION

1.1 INTRODUCTION

The study of Solitons as a stable particle has caught the imagination of Mathematicians, Condensed Matter Physicists, Biologists, Information Scientists, Economists, Stock Theory analysts, Oceanographers, Geologists to name just a few. Solitons are solitary waves which are known to propagate for very large distances in nonlinear media. Solitons were first observed in the year 1834 by a young Scottish engineer, John Scott Russell [108] while observing the movement of a barge in a canal. The problem before the mathematicians was how to account for the observation of John Scott Russell. What is important is that wave equations known at that time did not admit Soliton solutions. Thus it was particularly important to derive a wave equation which admits Solitary wave Solutions. In 1895 Kortweg de Vries, a Dutch mathematician derived the equation [42] of propagation of waves in a shallow water canal. These equations were nonlinear which came to be known as KdV equations admit Soliton solutions. In 1955, Fermi, Pasta, and Ulam [96] were investigating the energy distribution in a linear system with a nonlinear term added as a perturbation. In the absence of the perturbation the energy in each normal mode would be constant. When the nonlinear perturbation is turned on it is found that the energy does not spread to all the modes but remains in the initial mode and a few nearby modes. Further the energy density of these nearby modes has an almost periodic nature. This finding was not just characteristic of the FPU system but all nonlinear differential equations. While in a linear system the energy density of a mode remains constant, in a nonlinear system the energy density of a mode has an oscillatory behavior. The focus now shifted to the role of the nonlinear term in the nonlinear differential equation, particularly in the KdV equation. Zabusky and Kruskal [97] performed a numerical study on the KdV equation. They found that in a certain range the third order derivative balances the nonlinearity. When this happened they found that the solution assumes the shape of the Soliton. Via numerical simulation Zabusky and Kruskal had unearthed a very crucial feature of nonlinear differential equations: the energy lost in propagation is compensated by the energy generated by the nonlinear term.

It was obvious from the results of FPU, Zabusky and Kruskal that to understand nonlinear differential equations one must look into the Fourier domain. The question was can one develop equations equivalent to the nonlinear differential equation. Zhakarov and Shabat [133] first found the coupled equations which was equivalent to the nonlinear differential equations. AKNS [3] showed that these coupled equations correspond to rotations in the potential space. The potential was in fact the solution of the nonlinear differential equation. In fact one of the results is that they constructed the solution of the nonlinear differential equation as Fourier summation of the scattered waves from the potential. This equation was of fundamental significance in the field. It immediately gave rise to the principle of causality for nonlinear systems: the Gelfand-Levitan equation [25].

It was at this point noted condensed matter physicists James Krumhansl and J. R. Schrieffer [80] wrote their seminal paper on Statistical Mechanics of One Dimensional Systems. Krumhansl and Schrieffer considered a one dimensional array of masses interacting via a double well potential. They showed that such a system admits tanh Solitons. Further they showed that the observed behavior of the central peak in NMR can be accounted for by Solitons. Schrieffer followed this by another landmark paper namely one on Poly-acetylene. This paper elegantly computed the effect of Solitons on the electronic band gap of Poly-acetylene. Solitons compress the lattice and produce mid-gap states which may or may not have spin and charge. When the mid gap state de- excites it produces a charged or uncharged Soliton with or without spin. These Soliton states have in fact been observed in Poly-acetylene and other systems. The effect of this paper on the experimentalists was immense. The central point was to exploit the coupling between mid gap states and Solitons. If via Solitons one can produce mid gap states then it stands to reason that in sandwich systems one can produce mid gap state in one system and upon de-excitation produce Solitons in another system. Further if the mid gap states are made to absorb energy (solar energy for example), upon de-excitation the energy can be stored in the form of Solitons in some other media such as Lithium Niobate from where it can be later extracted in the form of current. Via such sandwich materials one can store solar energy and use it later. It now became clear that Solitons and its physics had been understood and the focus had shifted to its applications. As pointed out earlier the concept of Solitons has found application in diverse areas. Biologists find Solitons in the DNA lattice [128]. These Solitons induce conformational changes when a protein approaches. This forms the basis of intra cellular communication. The progress of Solitons on a DNA lattice, their interactions can be described in terms of Feynman diagrams. There are thus minimum set Feynman diagrams for the sustenance of cellular life. On the other hand inhibition of Solitons in the DNA (such as by binding of a ligand to DNA) can thus bring about cellular death.

Similarly light propagates as Solitons in Optical Fibers [50,95]. Information in the form of packets is sent as Solitons through the Optical Fibers. As Solitons travel with the speed of light such networks offer very high speed connectivity and high bandwidth. Solitons thus became a better substitute for electrons in printed circuit boards with one added advantage: distance. This opened the way for distributed printed circuit boards connected via Optical fibers. Applications of Solitons thus started appearing in almost every field. Oceanographers found that Tsunamis are Solitons as a result they propagate incredibly large distances (8000 km) without change of shape. In this thesis we concentrate on only one aspect: Nonlinear Differential Equations and their Soliton Solutions. The chapters of the thesis illustrate the various features of Nonlinear Differential equations.

1.2 AIMS AND OBJECTIVES

1.2.1 To find numerically Soliton solution of Nonlinear Schrodinger equations

1.2.2 To find long wave length Soliton solution of Navier Stokes equations.

1.2.3 To find the long wave length Soliton solution for the KdV equations.

1.2.4 To find Soliton solution of Sine Gordon equations in the long wavelength limit

1.2.5 To solve the Sine Gordon equation (unperturbed and perturbed)

1.2.6 To find Soliton solution in nonlinear Optical lattices.

1.2.7 To find Soliton solutions in Long Josephson junctions in a magnetic Field

1.2.8 To find the Vortex Soliton solution in Poly-acetylene.

1.3 EXPECTED OUTCOME

1. We can apply the method of Sakaguchi and Malomed to various nonlinear differential equations and find Solitons Solution in Long Wave length limit.

2. Appropriate redefinition of variable may be required to obtain the equivalent Schrodinger equation (as in Navier Stokes Equation). In this form the array of infinite conservation laws are applicable.

3. In the case of the nonlinear optical lattice we observed that the Lagrangian of the one dim nonlinear optical lattice corresponds to that of a double well potential, hence following J. A. Krumhansl and J. R. Schrieffer, we postulated the presence of tan hyperbolic domain wall solitons in nonlinear optical lattices. This conjecture is also verified via experimental result.

4. In Poly-acetylene cis and trans state represents diametrically opposite orientation and hence are separated by Soliton like profiles (Su, Schrieffer and Heeger (SSH))

5. This study will present a systematic method of finding Soliton Solutions (via both analytical and numerical methods) of certain nonlinear differential equations (both perturbed and unperturbed) in the long wavelength limit.

1.4 MATERIAL AND METHODS.

Discretized nonlinear differential equations (KdV and nonlinear Schrodinger equation) have been solved by many authors. In this paper we have solved nonlinear differential equations via lattice discretization. This is done by first removing the time dependence of the differential equation and then invoking lattice discretization [123]. This results in a difference equation. For nonlinear equations, lattice discretization results in a nonlinear difference equation which is solved by first taking a Z transform, then taking the inverse Z transform we get the solution. **A MATLAB program has been written which solves the difference equation and plots the solutions from the recurrence relation itself.** Via this technique we have solved Nonlinear Schrodinger equation, Toda Lattice equation of motion and Klein-Gordon equation. In each of these cases we have obtained Soliton solutions [Chapter 3].

Sakaguchi and Malomed [110] have obtained long wavelength expansion of the Gross-Pitaevskii equation. By following this method in the case of KdV equation we obtain (comparing coefficients) the effective equations in the long wavelength region [Chapter 5]. The effective equation is put in the form of a conservation law (using the technique of Zabusky and Kruskal [131]. The spatial component of the conservation law is an eigenvalue equation. For the KdV equation the eigenvalue equation is the Schrodinger equation. (Note that for Sine-Gordon, Navier–Stokes some variable transformations or averaging of variables over both space and time are required to obtain the equivalent Schrodinger equation.) **The important point is that in the asymptotic limit (we find) the nonlinear differential equation becomes equivalent to the Schrodinger equation**. In each case the Schrodinger equation is solved and the Green’s function obtained from the eigen-functions. We extend this technique to Sine Navier Stokes [Chapter 4] and Gordon [Chapter 6], equations. In each case we find bound states of the system.

From the Gross-Pitaevskii equation for nonlinear optical lattice we derive the Hamiltonian for the double well model. This model is solved to obtain tanh domain wall soliton solutions which have been observed and also derived by other authors using a different model. The domain wall soliton solutions predict lattice compressibility which has been observed. [Chapter 7]

Recently [114] have found spectacular series of phase jumps in electrons passing through a Josephson junction in a magnetic field. We propose that these jumps occur due to electrons escaping from a potential well formed by a kink anti kink pair and crossing the Josephson junction [Chapter 8]. Josephson junction is governed by Sine Gordon Equation. We first solve the Sine Gordon equation in the long wavelength limit following the technique first outlined by Sakaguchi and Malomed

[110] in their classic paper. Via this technique we find the Green's function in the long wave length limit. This agrees very well with Greens functions computed intuitively with approximate Green's functions of electrons in Josephson junctions. This therefore establishes that the approach adopted here is indeed correct. Thereafter one computes the bound states of the kink anti kink pair. Thereafter one uses the fact that bound states decay. In other words the electron escapes from the kink anti kink potential. The Gelfand-Levitan equation is applied to this process to obtain the phase jumps.

Solitons in conducting polymers have been attracting considerable attention in recent years. In these conducting polymers, Poly-acetylene is a particularly important example. Each carbon atom in a pure trans-polyacetylene contributes only a single $p\pi$ electron, and π band is only half full. We find vortex Soltons in Poly-acetylene [chapter 9]

Ablowitz, Kaup, Newell and Segur (AKNS) and also Zhakarov and Shabat [133] have shown that both temporal and spatial evolution equations are associated with a nonlinear differential equation. These evolution equations are linear Eigenvalue equations. ***For the unperturbed Sine-Gordon equation, the spatial evolution equation may be interpreted as a rotation in the potential space.*** In fact this conclusion should not come as a surprise as the Quantum Mechanical Schrodinger equation can be interpreted as a rotation in the Hilbert space. Hence the time evolution of the equivalent Schrodinger equation should be given by an appropriate rotation in the corresponding Hilbert space. This precisely is the finding of AKNS. In fact this angle of rotation in the potential space can be computed from the equivalent Schrodinger equation picture. And temporal evolution is the same as a rotation matrix in potential space through an angle u. We solve for the Eigen values for this rotation operator. **The Eigen values, which are in the form of operators, are solved**. Via this technique we will solve the perturbed Sine-Gordon equation. In the small amplitude limit we recover the kink solution of the Sine-Gordon equation implying that the operator approach employed here is correct [Chapter 10]**. This result implies that volution of certain nonlinear differential equation can be thought of as a transformation of the potential space.**

1.5 IMPORTANCE OF THE PRESENT WORK.

- In recent years, there has been an increased interest in the study of nonlinear phenomena. These systems exist in all research fields, such as fluid mechanics, elastic media, optical fibers, nuclear physics, relativistic quantum mechanics, high-energy physics, plasma physics, biology, solid-state physics, chemical kinematics, chemical physics, geochemistry, etc. Emphasis is put particularly on seeking explicit solutions for non-linear evolution equations. A variety of methods were devoted to studying different types of nonlinearities of optical materials and they have played (and still play) a major role in mathematical physics. In fact, scholars have been interested in the study of localized structure of waves, which ignore dispersion or diffraction processes. We will only highlight the importance of **Solitons** in real life where the nonlinearity plays an important role. It is said that soliton is a bridge between Mathematics and Physics
- The method is also applied to Long Josephson junctions in a magnetic field.
- Optical Solitons are an interesting subject in optical fiber communication because of their capability of propagation over long distance without attenuation and changes in shapes.

1.6 MOTIVATION.

The motivation for this work came from our efforts to understand the Soliton and its interactions. H. Sakaguchi and B.A Malomed [111] in their treatment of the Gross-Pitaevskii equation had developed a very interesting approach to obtain the long wavelength solution. We extended this approach to other nonlinear differential equations. Further H. Sakaguchi and B.A Malomed converted the nonlinear differential equation for nonlinear optical lattices into an equivalent Lagrangian. We noticed that this Lagrangian was that same as that of a double well potential and hence would admit domain wall solutions. This in fact was the case.

The spatial and temporal evolution equations corresponding to Sine Gordon and Sinh Gordon equation were interpreted as rotations in the potential space. This gave the idea that these operators can be solved for eigenvales in terms of the operators. The resulting operator equation can then be solved. It turned out that this was indeed possible. Nonlinear optical crystals are extremely fascinating. The underlying materials of these crystals are highly polarizable which respond to the electric fields of incident laser light in a time scale of attoseconds ($10^{-18} s$) It is this attosecond response time which has allowed us to propose that these crystals can be used as very fast optical switches Now this polarization induced by the incident laser light propagates in the material as a Soliton and it can be modulated. We note that Solitons (which are essentially waves) have both spin and charge . This has been first elaborated by Schrieffer et al. [119] This feature allowed us to propose the concept of very fast optical switches and massively parallel computing. Solitons are known to exist in the DNA. We identify Solitons with jumping genes and suggest that they are responsible for onset of cancer.

1.7 LIST OF NONLINEAR DIFFERENTIAL EQUATIONS USED IN THIS THESIS

1. KdV Equation

$$\psi_t + \psi_{xxx} - 6\psi\psi_x = 0$$

2. MKdV Equation

$$\psi_t + \psi_{xxx} \pm 6\psi^2\psi_x = 0$$

3. Nonlinear Schrodinger Equation (NLSE)

$$i\psi_t + \psi_{xx} + \psi|\psi|^2 = 0$$

4. Sine Gordon Equation.

$$\psi_{tt} - \psi_{xx} + \sin\psi = 0$$

5. Phi four Equation

$$\psi_{tt} - \psi_{xx} - \psi + \psi^3 = 0$$

6. Navier Stokes Equation

$$\rho\left(\frac{\partial v}{\partial t} + v.\nabla v\right) = -\nabla p + \nabla.T + f$$

7. Toda Lattice:

$$\ddot{\Phi}(n)=\exp(\Phi(n-1)\text{-}\ \Phi(n))\text{-}\exp\ (\Phi(n)\text{-}\ \Phi(n+1)$$

8. Klein-Gordon equation

$$\mathbf{\Phi}_{\mathbf{xx}} - \mathbf{\Phi}_{\mathbf{tt}} = \mathbf{m}^2\mathbf{\Phi}$$

9. The perturbed Sine-Gordon equation.

$$\psi_{tt} - \kappa\psi_{xx} + \sin\psi = -\alpha\psi_t + \beta\psi_{xxt} - \gamma + \varepsilon\varsigma + F$$