CHAPTER 10

SOLITON SOLUTION OF THE UNPERTURBED SINE-GORDON AND PERTURBED SINE-GORDON EQUATION

10. 1 INTRODUCTION:

In this chapter we find the solution of the Unperturbed Sine-Gordon And Perturbed Sine-Gordon Equation. Ablowitz, Kaup, Newell and Segur (AKNS) and also Zhakarov and Shabat (ZS) have shown that both temporal and spatial evolution equations are associated with a nonlinear differential equation. These evolution equations are linear Eigenvalue equations. For the unperturbed Sine-Gordon equation, the spatial evolution equation may be interpreted as a rotation in the potential space. And temporal evolution is the same as a rotation matrix in potential space through an angle u. We solve for the Eigen values for this rotation operator. The Eigen values, which are in the form of operators, are solved. Via this technique we solve the perturbed Sine-Gordon equation. In the small amplitude limit we recover the kink solution of the Sine-Gordon equation implying that the operator approach employed here is correct. This result implies that evolution of all nonlinear differential equation can be thought of as a transformation of the potential space.

10.2. PERTURBED SINE-GORDON EQUATION

The perturbed Sine-Gordon equation is [74]

$$u_{tt} - \kappa u_{xx} + \sin u = -\alpha u_t - \gamma + \varepsilon\varsigma \tag{1}$$

Where x ∈R, α ≥ 0, γ ∈R are constants ε is a small parameter and depends upon various external variables. The terms on the right hand side is zero we get the (unperturbed) Sine-Gordon equation. In view of its importance, the perturbed Sine-Gordon equation, has been studied by various authors [37,40, 24]. To obtain a solution for ($\varepsilon \neq 0$) one first solves the ($\varepsilon = 0$) equation. The solution of the Sine-Gordon equation is a kink. In a multiple scale analysis [98, 5], one expands the solution u as

$$u = u_0 + u_1 \tag{2}$$

where u_0 is the original solution (kink solution) of the Sine-Gordon equation and u_1 provides a dressing term. However the major difference between D.J Kaup and El-sayed Osman [73, 74] and multiple scale analyses [98, 5] is that [74] convert the equation for u_1 into an eigenvalue equation. The solution of the complete equation is obtained in terms of these eigenvalues. As the approach of [74] is more general we opt for this approach in finding the approximate solution of the perturbed Sine-Gordon equation.

In a classic paper M. Ablowitz, D. Kaup, A. Newell, A. Segur [9, 4], hereafter AKNS, developed a unique method of solving nonlinear differential equations. In their method, one has to solve eigenvalue equations corresponding to the nonlinear differential equations. The solution to the nonlinear differential equation is the potential in the eigenvalue equations. Both AKNS [9, 4] and also ZS [81] have shown that both temporal and spatial evolution equations are associated with a nonlinear differential equation. These evolution equations are linear eigenvalue equations. For the unperturbed Sine-Gordon equation, the spatial evolution equation has operators. We solve for the eigenvalues for this operator equation. The eigenvalues, which is in the form of operators, is now solved.

10.3 EIGENVALUES OF THE SPATIAL EVOLUTION EQUATION

The ZS equations (in case of perturbed Sine- Gordon) are

$$\begin{aligned} \varphi_{1x} + i\zeta\varphi_1 &= q(x,t)\varphi_2 \\ \varphi_{2x} - i\zeta\varphi_2 &= r(x,t)\varphi_1 \end{aligned} \tag{3}$$

where the potentials $q(x,t), r(x,t)$ are taken as usual to be rapidly decaying smooth functions. To obtain the Sine-Gordon equation from (3) we take the transformations as taken by AKNS [3].

$$q \to -\frac{1}{2}u_x, r \to \frac{1}{2}u_x \tag{4}$$

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}_x = \begin{pmatrix} -i\zeta & -\frac{u_x}{2} \\ \frac{u_x}{2} & i\zeta \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{5}$$

This may be written as

$$\boldsymbol{\phi}_x = \boldsymbol{T\phi} \tag{6}$$

where
$$T = \begin{pmatrix} -i\zeta & -\frac{u_x}{2} \\ \frac{u_x}{2} & i\zeta \end{pmatrix} \tag{7}$$

and
$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{8}$$
is called the state vector

Similarly the temporal evolution is given by

$$\phi_t = R(u)\phi \tag{9}$$

where
$$R(u) = \frac{i}{4\zeta} \begin{pmatrix} \cos(u) & \sin(u) \\ \sin(u) & -\sin(u) \end{pmatrix} \tag{10}$$

where $R(u)$ is the rotation matrix in the u space by angle u . Temporal evolution simply rotates the state vector φ by the angle u. Rotation in the u space generates the time derivative. This is analogous to quantum mechanics where the time evolution of a quantum state can be viewed as rotation in the Hilbert space [80]. As both T and R (u) operators are in the Hilbert space they can be written as eigenvalue equations namely

$$T\phi = \lambda_x \phi \tag{11}$$

$$R(u)\phi = \lambda_t \phi \tag{12}$$

Further, we look for travelling wave solutions of the type $f(x - vt)$. This implies

$$\frac{\partial^2}{\partial t^2} = v^2 \frac{\partial^2}{\partial x^2} \tag{13}$$

By differentiating (11), (12) and using (6) we obtain

$$\phi_{tt} = \lambda_x^2 v^2 \phi \tag{14}$$

We solve equation (11) for the Eigen value λ_x. The Eigen value λ_x is in terms of operators u_x. We solve the operator equation and substitute in (20). Equation (20) is then solved using approximate methods developed by Kaup and El-sayed Osman [73, 74]. The Eigenvalues λ_x are given by

$$\lambda_x = \frac{\pm\left(4\zeta^2 - u_x^2\right)^{1/2}}{2} \tag{15}$$

Now (11) may be written as

$$\phi_x = \lambda_x \phi \tag{16}$$

The solution of this equation is

$$\phi = \phi_0 e^{\lambda_x x} \tag{17}$$

Taking the log of both sides we obtain

$$\ln\frac{\phi}{\phi_0} = \lambda_x x \tag{18}$$

Using the value of λ_x in (6) and simplifying we obtain

$$\frac{du}{dx} = 2\left(\zeta^2 - \frac{1}{x^2}\left(\ln\frac{\phi(u)}{\phi_0(u)}\right)^2\right)^{1/2} \tag{19}$$

which gives the integral

$$x = \int_o^u \frac{du}{\left(\zeta - \frac{1}{x}\left(\ln\frac{\phi(u)}{\phi_0(u)}\right)\right)^{1/2}\left(\zeta + \frac{1}{x}\left(\ln\frac{\phi(u)}{\phi_0(u)}\right)\right)^{1/2}} \tag{20}$$

The solution of (20) is [128]

$$u = a\,\mathrm{sn}(b\phi) \tag{21}$$

Note that for low amplitudes sn($b\phi$) = sin($b\phi$) For larger amplitudes one has sn($b\phi$) = tanh($b\phi$). We confine ourselves to the large amplitude limit.

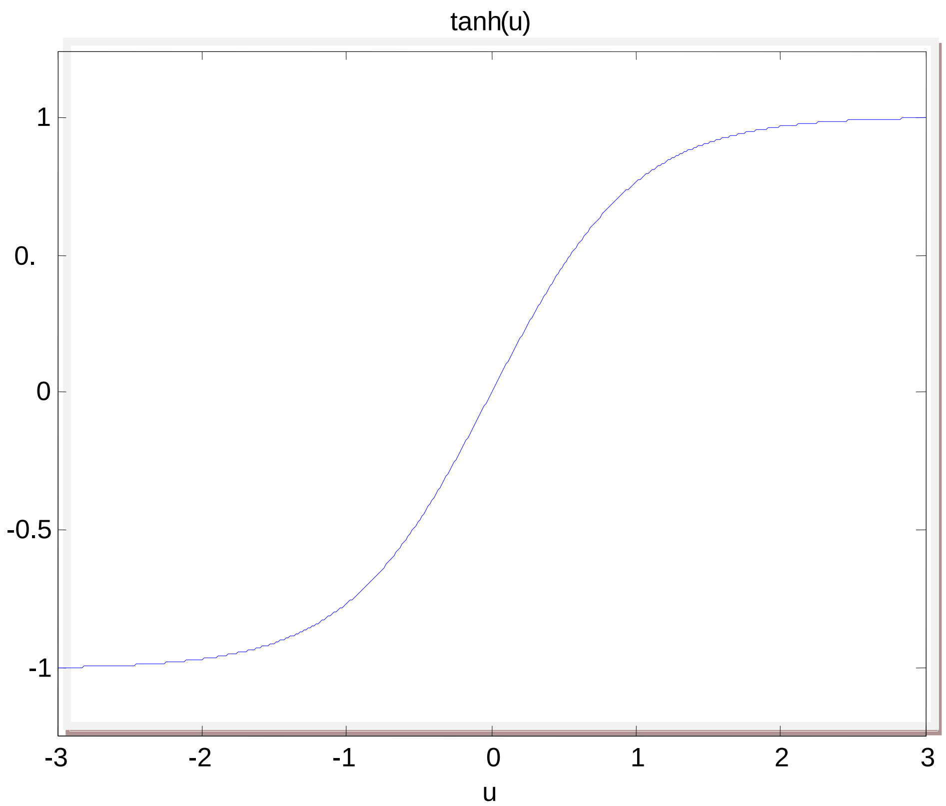

Figure 1

where $\mathrm{sn}\left(b\phi\right)$ is the elliptic sine function and ϕ is determined in the next section. Note that this is the inverse rotation which relates the potential to the state vector ϕ. We find that the solution of the perturbed Sine-Gordon is

$$\eta = a\,\mathrm{sn}\left(b\exp\left(\frac{4\lambda_x}{\kappa^2\left(v_0^{(1)}\right)^2}(x-v_0t)^2\right)\right) \tag{22}$$

In the small amplitude limit, we find

$$\eta = a\sin s \tag{23}$$

where

$$s = \left(b\exp\left(\frac{4\lambda_x}{\kappa^2\left(v_0^{(1)}\right)^2}(x-v_0t)^2\right)\right) \tag{24}$$

We note that for small η, s

$$\tan(\eta) \approx \eta,\ \sin(s) \approx s \tag{25}$$

Using (25) in (23) we find

$$\tan(\eta) = s \quad \tan(\eta) = \sin(s)$$

$$\text{or}\ \eta = \arctan\left(\sin s\right) \tag{26}$$

Hence the solution become (for small s)

$$\eta = \arctan\left(\exp\left(\frac{4\lambda_x}{\kappa^2 \left(v_0^{(1)}\right)^2}(x - v_0 t)^2\right)\right) \tag{27}$$

In the small amplitude limit we recover the kink solution of the Sine-Gordon equation. We note that the functional form of (27) is different from the kink solution of the unperturbed Sine-Gordon equation. However for the solution to be valid for larger amplitudes function we have to use the elliptic function solution.

10.4 PERTURBATION EXPANSION

In this section we determine the temporal evolution of the state vector ϕ using the Perturbation methods of [74], We now use the results of (21) in (20) to obtain

$$\phi_{tt} = \lambda_x^2 v^2 \phi \qquad = \frac{v^2}{x^2}\left(\ln\left(\frac{\phi}{\phi_0}\right)\right)^2 \phi \tag{28}$$

Put $$\frac{\phi}{\phi_0} = \exp\left(\kappa u x\right) \tag{29}$$

in (28) to obtain

$$\ln\left(\frac{\phi}{\phi_0}\right) = \kappa u x,\ \left(\ln\left(\frac{\phi}{\phi_0}\right)\right)^2 = \kappa^2 u^2 x^2 \tag{30}$$

Hence

$$\phi_{tt} = v^2 \kappa^2 u^2 \phi_0 \exp\left(\kappa u x\right) \tag{31}$$

Now we wish to obtain express (31) in terms of u as the external forces will be in terms of u. Differentiating (29) and substituting in (31) we obtain $\kappa^2 u_t^2 + \kappa u_{tt} - v^2 \kappa^2 u^2 = 0$ (32)

Following [96] we expand

$$u(x,t) = u_0(x) + \varepsilon u_1(\chi,\tau) + \varepsilon^2 u_2(\chi,\tau) + ...\qquad(33)$$

where $\chi = \chi_0 + \varepsilon_1 \chi_1(t) + \varepsilon^2 \chi_2(t) + ...\qquad(34)$

$$\begin{aligned}\tau &= t\\ \chi_0 &= x - v_0 t\end{aligned}\qquad(35)$$

where v_0 is the zero order velocity of kink. For perturbation theory, it is convenient to replace the temporal derivatives by spatial derivatives to obtain

$$\kappa^2 v_0^2 u_{0\chi}^2 + \kappa v_0^2 u_{\chi\chi} - v^2\kappa^2 u_0 = 0\qquad(36)$$

Using the expansions (33), we obtain zeroeth order equation from (36) as

$$\kappa v_0^2 u_{0\chi}^2 + v_0^2 u_{\chi\chi} = u(u_0)\qquad(37)$$

The first order equation is given by

$$\begin{aligned}&2u_{0\chi}\left[u_{1\chi\chi} - v_0 u_{1\chi t} + v_0 u_{1t\chi} + u_{1tt} v_0^2\right] +\\ &\kappa\left[v_0^2 u_{1\chi\chi} + u_{1\chi t} - v_0 u_{1t\chi} + u_{1tt}\right] - v^2\kappa^2 2u_0 u_{1\chi} = 0\end{aligned}\qquad(38)$$

We expand the kinks velocity as

$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + ...\qquad(39)$$

Further we expand each u_i, v_i in a series as follows

$$u_i = u_i^{(0)} + Ru_i^{(1)} + ...i = 0,1,2...\qquad(40)$$

$$v_i = v_i^{(0)} + Rv_i^{(1)} + ...i = 0,1,2...\qquad(41)$$

Using the Perturbation expansion

for the zeroth order we get $\kappa^2\left(v_0^{(1)}\right)^2\left(u_{0\chi}^{(1)}\right)^2 + \kappa v_0 u_{\chi\chi}^{(1)} = uu_0^{(1)}\qquad(42)$

Define the operator $L^{(0)} = -\kappa v_0 \partial_{\chi\chi} + u\qquad(43)$

Then the eigenvalue equation becomes

$$L^{(0)} u_0^{(1)} = \kappa^2\left(v_0^{(1)}\right)^2\left(u_{0\chi}^{(1)}\right)^2\qquad(44)$$

Now (42) may be written us $\begin{aligned}&\kappa^2\left(v_0^{(1)}\right)^2\left(u_{0\chi}^{(1)}\right)^2 = L^{(0)} u_0{}^{(1)} = \lambda\, u_0{}^{(1)}\\ &\phi = \phi_0 \exp\left(\kappa u_0^{(1)} x\right)\end{aligned}\qquad(45)$

On simplifying (45) we get $u_0^{(1)} = \frac{4\lambda}{\kappa^2 \left(v_0^{(1)}\right)^2} \chi^2$ (46)

Now from (29), on using the perturbation expansion we get

$$\phi = \phi_0 \exp\left(\kappa u_0^{(1)} x\right) \tag{47}$$

Using (46) in (47) we obtain $\phi = \phi_0 \exp\left(\frac{4\lambda_x}{\kappa^2 \left(v_0^{(1)}\right)^2} (x - v_0 t)^2 \right)$ (48)

10.5 CONCLUSION

Both AKNS and ZS have shown that with a nonlinear differential equation can be associated with a spatial and temporal evolution equation. For the unperturbed Sine-Gordon equation the temporal evolution equation corresponds to a rotation in the potential space. We determine the Eigenvalues of the spatial evolution operator (6) in terms of the operators u_x. The equation is thus solved to obtain the potential in terms of the state vector. By requiring travelling wave solution of the form $f(x - vt)$ equation (20) is solved via the perturbation expansion of [40] to obtain the time evolution of the state vector. This technique can be generalized to obtain the solution of other nonlinear differential equations.