CHAPTER 8

LONG JOSEPHSON JUNCTIONS IN MAGNETIC FIELD

8.1. INTRODUCTION

Recently have found spectacular series of phase jumps in electrons passing through a Josephson junction in a magnetic field. We propose that these jumps occur due to electrons escaping from a potential well formed by a kink anti kink pair and crossing the Josephson junction. Josephson junction is governed by Sine Gordon Equation. We first solve the Sine Gordon equation in the long wavelength limit following the technique first outlined by Sakaguchi and Malomed in their classic paper. Via this technique we find the Green's function in the long wave length limit. This agrees very well with Greens functions computed intuitively with approximate Green's functions of electrons in Josephson junctions. This therefore establishes that the approach adopted here is indeed correct. Thereafter one computes the bound states of the kink anti kink pair. Thereafter one uses the fact that bound states decay. In other words the electron escapes from the kink anti kink potential. The Gelfand-Levitan equation is applied to this process to obtain the phase jumps.

We develop a model to account for the recently observed phase jump of electrons in Josephson Junction, in a magnetic field, as the electrons cross the junction. We suggest that electrons are trapped in the potential formed by a kink anti-kink pair. When the electron escapes from this potential well it suffers a potential jump as it crosses the junction. Electrons at lower depths suffer greater potential jumps. The potential jumps were evaluated by using the Lax pair for the Sine Gordon equation and then using Gelfand-Levitan equation on the bound states formed by the kink-anti kink pair.

8.2 LATTICES

Lattice model is a physical model that is defined on a lattice, as opposed to the continuum of space or space-time. Lattice models originally occurred in the context of condensed matter physics, where the atoms of a crystal automatically form a lattice. Currently, lattice models are quite popular in theoretical physics, for many reasons.

Some models are exactly solvable, and thus offer insight into physics beyond what can be learned from perturbation theory. Lattice models are also ideal for study by the methods of computational physics, as the discretization of any continuum model automatically turns it into a lattice model. Examples of lattice models in condensed matter physics include the Ising model, the Potts model, the XY model, the Toda lattice. The exact solution to many of these models (when they are solvable) includes the presence of solitons. Techniques for solving these include the inverse scattering transform and the method of Lax pairs, the Yang-Baxter equation and quantum groups. The solution of these models has given insights into the nature of phase transitions, magnetization and scaling behavior, as well as insights into the nature of quantum field theory. Physical lattice models frequently occur as an approximation to a continuum theory, either to give an ultraviolet cutoff to the theory to prevent divergences or to perform numerical computations. An example of a continuum theory that is widely studied by lattice models is the QCD lattice model, a discretization of quantum chromo dynamics. More generally, lattice gauge theory and lattice field theory are areas of study. Lattice models are also used to simulate the structure and dynamics of polymers. Examples include the bond fluctuation model and the 2nd model

8.3 JOSEPHSON JUNCTION

Josephson junction has been studied by a number of authors [66, 11, 48, 67]. Further Solitons in Josephson junctions has been both predicted [84, 12, 85] and found experimentally [84, 12]. Josephson Junctions are described by Sine Gordon equation which has kink Soliton solutions.

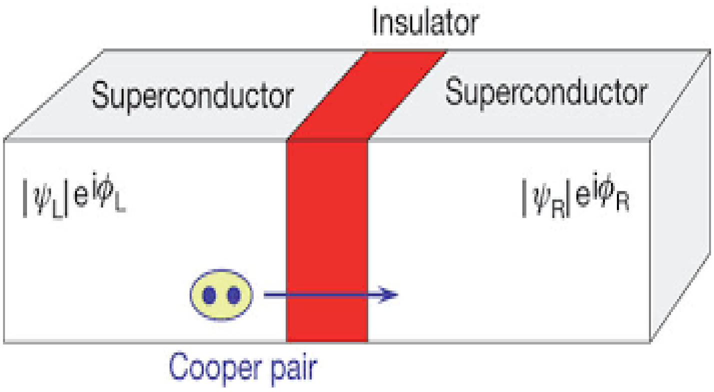

Figure 1.

JOSEPHSON JUNCTIONS

8.4 PHYSICAL BASIS

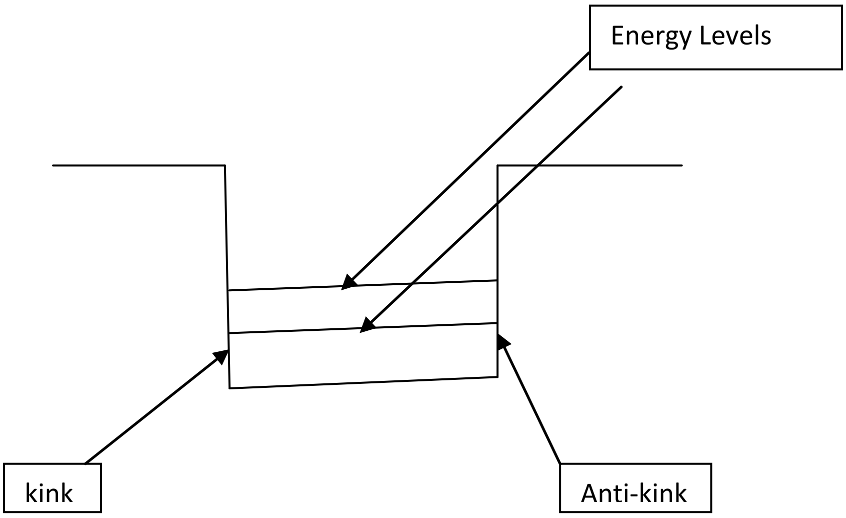

Figure 2

These Solitons tunnel through the Josephson junction barrier. A detailed numerical analysis of Josephson tunnel junctions has been done by Lomdahl, Soerensen and Christiansen [85]. They find comprehensive numerical evidence of Solitons in both long and intermediate junctions. Charge Soliton Solutions have been found by Ziv Herman, Eshel Ben-Jacob and Gerd Schon [135] for serially coupled Josephson junctions. T. Doderer et. al. [41] have experimentally stimulated Solitons in Josephson junctions and studied their dynamics. They find that the junction properties are accurately described by the perturbed Sine Gordon equation.

Recently [120] have found spectacular series of phase jumps in electrons passing through a Josephson junction in a magnetic field. We propose that these jumps occur due to electrons escaping from a potential well formed by a kink anti kink pair and crossing the Josephson junction. We first solve the Sine Gordon equation in the long wavelength limit following the technique first outlined by Sakaguchi and Malomed [61] in their classic paper. Via this technique we find the Green's function in the long wave length limit. This agrees very well with Greens functions computed intuitively with approximate Green's functions of electrons in Josephson junctions. This therefore establishes that the approach adopted here is indeed correct. Thereafter one computes the bound states of the kink anti kink pair. Thereafter one uses the fact that bound states decay. In other words the electron escapes from the kink anti kink potential. The Gelfand-Levitan equation is applied to this process to obtain the phase jumps.

8.5 SOLUTION OF THE SINE GORDON EQUATION IN ASYMPTOTIC LIMIT

The Sine –Gordon equation is

$$\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}+\sin\psi=0 \tag{1}$$

We look for solutions of the form [111]

$$\psi\left(x,t\right)=\psi^{(0)}\left(x,t\right)+\psi^{(1)}\left(x,t\right)\cos\left(2x\right)+.... \tag{2}$$

Using

$$e^{ixsi\,\mathrm{n}\,\phi}=\sum_{n=-\infty}^{n=+\infty}J_n\left(x\right)e^{in\phi} \tag{3}$$

Equating the coefficients of cos2x

$$\psi_{tt}^{(0)}=\psi_{xx}^{(0)}-\sum_{n=-\infty}^{n=+\infty}\left[\sin\psi^{(0)}\cos n\left(\phi+\frac{\pi}{2}\right)\cos\psi^{(0)}\sin n\left(\phi+\frac{\pi}{2}\right)\right]J_n\left(\psi^{(1)}\right) \tag{4}$$

For n=0

$$\psi_{tt}^{(0)}=\psi_{xx}^{(0)}-\left[\sin\psi^{(0)}J_0\left(\psi^{(1)}\right)\right] \tag{5}$$

Assume a travelling wave solution

$$\psi^{(0)}=f\left(x-vt\right), \tag{6}$$

$$\psi_{tt}^{(0)}=v^2 f'' \quad \sin\psi^{(0)}\approx\psi^{(0)} \tag{7}$$

On substituting above we get

$$v^2 f''=f''-J_0\left(\psi^{(1)}\right)f \tag{8}$$

$$\frac{f''}{f}=\frac{J_0\left(\psi^{(1)}\right)}{1-v^2} \tag{9}$$

The solution of (25) must be of the form

$$\psi^{(0)}=\exp\left((x-vt)\sqrt{\frac{J_0\left(\psi^{(1)}\right)}{1-v^2}}\right) \tag{10}$$

We now derive the conservation equation corresponding to (21). Using

$$\psi_t^{(0)}=\phi,\psi^{(0)}=\frac{\partial\rho}{\partial x}, \tag{11}$$

we obtain

$$\phi_t=\rho_{xxx}-\sin(\psi^{(0)})J_0(\psi^{(1)}) \tag{12}$$

Using the asymptotic expansion of $J_0(\psi^{(1)})$

$$J_0(\psi^{(1)})\approx\sqrt{\frac{2}{\pi\psi^{(1)}}}\cos\left(\psi^{(1)}-\frac{\pi}{4}\right) \tag{13}$$

We get

$$J_0(\psi^{(1)})\approx\frac{1}{\sqrt{\pi}}\left[\frac{1}{\sqrt{\psi^{(1)}}}+\sqrt{\psi^{(1)}}\right] \tag{14}$$

Using $\psi^{(1)}=e^{i\theta}$ (15)

In the θ space the eigen value equation is

$$\rho_{\theta\theta} + \left(\lambda - \frac{2}{\sqrt{\pi}} \cos(\psi^{0}) \cos\frac{\theta}{2} \right) \rho = 0 \tag{16}$$

Since we are interested in the asymptotic limit, we take the t =0 solution of (17) as the effective potential in (22). The equation to solve is

$$\rho_{\theta\theta} + \left(\lambda - \exp\left((x) \sqrt{\frac{J_0\left(\psi^{(1)}\right)}{1-v^2}} \right) \right) \rho = 0 \tag{17}$$

Using $\phi(s) = \int_0^\infty e^{-\theta s} \rho(\theta) d\theta$ (18)

Equation (17) becomes

$$(s^2 - \lambda)\phi(s) - \phi(s+k) = s\phi(0) + \phi^{'}(0) \tag{19}$$

OR

$$\phi(s) = \frac{\phi(s+k)}{s^2-\lambda} + \frac{s\phi(0)}{s^2-\lambda} + \frac{\phi^{'}(0)}{s^2-\lambda} \tag{20}$$

Taking the inverse transform

$$\rho(\theta) \square \frac{\phi^{'}(0)\cos\sqrt{\lambda}\theta}{1-e^{\sqrt{\lambda}\theta}\cos\sqrt{\lambda}\theta} \tag{21}$$

Using $\phi(0) = 0$

When $e^{\sqrt{\lambda}\theta} \cos\sqrt{\lambda}\theta \square\ 1$

$$\rho(\theta) \square\ \phi^{'}(0) e^{-\sqrt{\lambda}\theta} \tag{22}$$

The Green's function is

$$G(x,x') = \phi^{'}(0) \sum_{R}^{\infty} e^{-k(\theta-\theta')} \tag{23}$$

Eqn. (23) is the Green's function of an electron in a Josephson junction. The above result could have been derived intuitively by noting that the wave function of an electron in a Josephson junction is $\psi(\vec{r}) = e^{k\theta}$

The Green's function may now be written as

$$G(r,r^{'}) = \sum_{k} e^{k(\theta^{'}-\theta)} \tag{24}$$

Thus the result derived in (23) is in agreement with (24) derived from basic physical considerations.

8.6 LAX OPERATERS

The Lax operators for the Sine Gordon equation are

$$L = I\frac{\partial}{\partial x} + \frac{i}{2}\left[\lambda - \frac{\cos(u)}{4\lambda}\right]\sigma_3 - \frac{\sigma_2}{4\lambda}\sin(u) + \frac{\sigma_1}{2}\left(u_x - u_t\right) \; B = I\frac{\partial}{\partial x} + \frac{i}{2}\left[\lambda + \frac{\cos(u)}{4\lambda}\right]\sigma_3 + \frac{\sigma_2}{4\lambda}\sin(u) + \frac{\sigma_1}{2}\left(u_x - u_t\right) \quad (25)$$

Let $\phi(k,t)$ be a soliton solution of the Sine Gordon equation. Since the Soliton is a localized solution we must have

$$\phi(k,t) = a_+(k,t)e^{-ikx} \; as \; x \to \infty$$

$$\phi(k,t) = b\_(k,t)e^{+ikx} \; as \; x \to -\infty \quad (26)$$

Now the time evolution of $\phi(k,t)$ is given by

$$i\frac{\partial \phi(k,t)}{\partial t} = B(t)\phi(k,t) \quad (27)$$

Assuming the operator B is time independent we obtain

$$\phi(k,t) = e^{-iBt}\phi(k,0) \quad (28)$$

where

$$B = h_0 1 + h_1\sigma_1 + h_2\sigma_2 + h_3\sigma_3 \quad (29)$$

$$h_0 = \frac{\partial}{\partial x}, h_1 = \frac{(u_x - u_t)}{2}, h_2 = \frac{\sin(u)}{4\lambda}, h_3 = \frac{i}{2}\left(\lambda + \frac{\cos(u)}{4\lambda}\right) \quad (30)$$

and

$$u = \tan^{-1}\left(\exp\left(\frac{x - vt}{\sqrt{1 - v^2}}\right)\right) \quad (31)$$

where u is the kink solution of the Sine Gordon equation. Using (37) – (39) we obtain

$$a_+(k,t) = e^{-it(h_0 + h_3)}a_+(k,0)$$

$$b\_(k,t) = e^{-it(h_0 - h_3)}b\_(k,0) \quad (32)$$

8.7 BOUND STATES OF THE KINK-ANTIKINK

Kink and anti kink form a potential well which can be approximated by a harmonic oscillator type of well. Such a well will have bound states. Let $\psi_n(x,0)$ be the bound state . Now the bound state wave function satisfies the boundary conditions

$$\psi_n(x,0)=\begin{Bmatrix} R_n(0)e^{-K_n x} & x\to\infty \\ T_n(0)e^{K_n x} & x\to-\infty \end{Bmatrix} \tag{33}$$

where $R_n(0)$ and $T_n(0)$ are normalization constants. The time evolution of the bound state wave function is given by

$$\psi_n(x,t)=e^{-iK_n t}e^{-h_3 t}\psi_n(x,0) \tag{34}$$

Note the replacement of k by iK_n. The normalization constant is

$$M_n(t)=e^{-2h_3 t}M_n(0) \tag{35}$$

This simple result tells us that the bound state decays exponentially in time – a fact that has been verified via numerous experiments.

8.8 GELFAND-LEVITAN EQUATION

In the inverse scattering method the Gelfand-Levitan equation is used to determine the scattering potential V(x,t) for all x,t. The scattering potential satisfies

$$V(x,t)=-2\frac{dg(x,x)}{dx} \tag{36}$$

where $g(x,y)$, for x < y, is the solution of the Gelfand-Levitan equation. Note that causality is built into the system via the inequality. The Gelfand-Levitan equation is

$$g(x,y)+K(x+y)+\int_x^\infty K(y+y)g(x,y)dy=0 \tag{37}$$

With

$$K(y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R(k,t)e^{iky}dk+\sum_{n=1}^{N}M_n e^{-K_n y} \tag{38}$$

To solve (38) we take $R(k,0)=0$ and the bound state energy as $-K^2$. We then obtain

$$g(x,y,t)+e^{-2h_3(x)t}M_n(0)e^{-K(x+y)} + \int_x^{\infty} e^{-2h_3(x)t}M_n(0)e^{-K(x+y)}g(x,y,t)dy=0 \quad (39)$$

Since we know that a bound state has an exponential decay we can write

$$g(x,y,t)=e^{-Ky}h(x,t) \quad (40)$$

We then obtain

$$g(x,y,t)=\frac{M_n(0)e^{-2h_3(x)t}e^{-K(x+y)}}{\left(1-e^{3Kx+Ky}M_n(0)e^{-2h_3(x)t}\right)} \quad (41)$$

Expanding the numerator one obtains

$$g(x,y,t)=M_n(0)e^{-2h_3(x)t}e^{-K(x+y)}\left(1+e^{3Kx+Ky}M_n(0)e^{-2h_3(x)t}+...\right) \quad (42)$$

Note that h_3 has been defined in (28). Each term in causes a phase jump. Phase jumps in the electron wave functions have recently been observed [120].

8.9 CONCLUSION

We have solved the Sine Gordon equation in the long wavelength approximation using the methods of Sakaguchi and Malomed [111]. The Greens function so obtained is found to agree with results obtained on the basis of wave functions of electrons in a Josephson junction. Now the Sine Gordon equation admits both kink and anti-kink solutions. A kink and anti-kink can form a potential well analogus to harmonic oscillator potential. An electron can get trapped in suc a well. We use the Gelfand –Levitan equation to find the amplitude for the electron to tunnel form kink-anti kink potential to a free state. The solution shows that there is phase jump in the wave function of the electrons as they tunnel through the junction. This phase jump has recently been observed.