CHAPTER 7

DOMAIN WALL SOLITON SOLUTION IN NONLINEAR OPTICAL LATTICES

7.1 INTRODCTION.

In this chapter based on the Lagrangian for the Gross –Pitaevskii equation as derived by H. Sakaguchi and B.A Malomed we have derived a double well model for the nonlinear optical lattice. This model explains the various features of nonlinear optical lattices. Further, from this model we obtain and simulate the probability for tunneling from one well to another which agrees with experimental results.

Lagrangian (L) = Kinetic Energy (T)-Potential Energy (V)

$L = T - V$

Based on the Lagrangian for the Gross –Pitaevskii equation as derived by H. Sakaguchi and B.A Malomed [111] we have derived a double well model for the nonlinear optical lattice. This model explains the various features of nonlinear optical lattices. Further, from this model we obtain and simulate the probability for tunneling from one well to another which agrees with experimental results [115].

Bose Einstein condensation has been both predicted and observed in harmonic oscillator potentials [78]. Nonlinear optical lattices have been known to simulate BEC via Feshbach Resonance [45]. It is therefore natural to assume a harmonic potential at each site of the nonlinear optical lattice. Further, Solitons have been predicted [19] and observed [87] in these lattices. We know from the seminal work of Krumhansl and Schrieffer [80] that a double well potential gives rise to domain wall Solitons. Since domain wall Solitons are indeed observed [102] in nonlinear optical lattices we suggest that a double well potential may model nonlinear optical lattices. Using the double well model for coupled nonlinear optical lattices we obtain Soliton solutions for one and higher dimensions. For the one dimensional lattice we find domain wall Solitons which induce lattice compression, which have been observed experimentally [115].

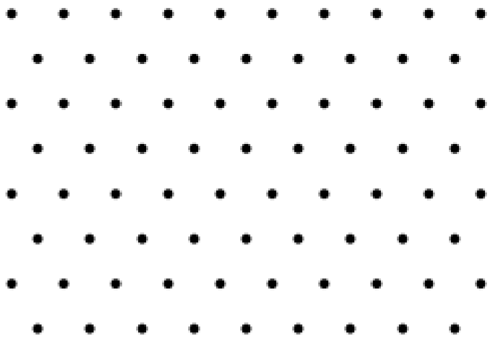

Figure 1 A lattice in the Euclidean Plane

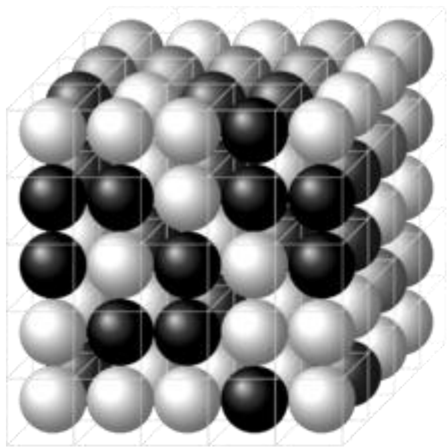

Figure 2

A three-dimensional lattice filled with two molecules A and B, here shown as black and white spheres. Lattices such as this are used - for example - in the Flory–Huggins solution theory

7.2 OPTICAL LATTICES

An **optical lattice** is formed by the interference of counter-propagating laser beams, creating a spatially periodic polarization pattern. The resulting periodic potential may trap neutral atoms via the Stark shift. Atoms are cooled and congregate in the locations of potential minima. The resulting arrangement of trapped atoms resembles a crystal lattice.

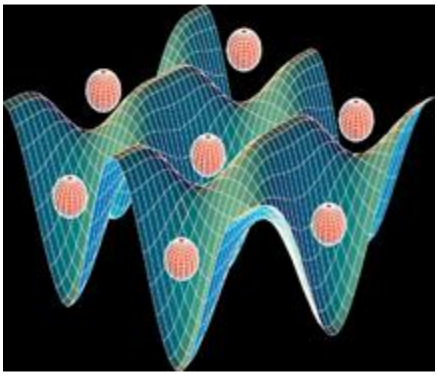

Figure 3

Simulation of an optical lattice potential.

Recently in the seminal paper of H. Sakaguchi and B.A Malomed [111] the Lagrangian corresponding to the Gross –Pitaevskii equation was derived. The potential in the Lagrangian is the double well potential which has been treated in the classic paper of Krumhansl and Schrieffer [80]. As shown in [80] the double well potential admits Domain wall Solitons. Further we find light and dark Solitons in the coupled lattice (but not in the uncoupled lattice) which is again verified by recent experiments on coupled lattices [27, 35].

7.3 MODEL

The Lagrangian corresponding to the Gross –Pitaevskii equation, derived in [111], is

$$L = \int_{-\infty}^{+\infty} \left[2\mu\phi^2 - \left(\frac{d\phi}{dx} \right)^2 + \left[\cos(2x) - g_0 \right] \phi^4 \right] dx \tag{1}$$

This represents a double well potential of the form

$$V(x) = \frac{Ax^2}{2} + \frac{Bx^4}{4} \tag{2}$$

where A = 4μ and B =4($Cos(2x) - g_0$). We note that for potential minima to be real either A< 0 or B< 0 (but not both). This implies either μ < 0 or ($Cos(2x) - g_0$) <0. Both conditions have been found in "(1)" for the existence of Solitons. We note from [87] that the height of the double well is given by $-\frac{1}{4}|A|^2 / B \quad = \frac{-\mu^2}{\cos(2x) - go}$ where $g_0 < Cos(2x)$ Note that we have two different regimes: (a) Height of the double well >> inter site interaction energy or (b) Height of double well << inter site interaction energy. The latter case (case (b)) correspond the occurrence of the Solitons. We find domain wall Solitons for the one dimensional lattice and lattice compression. However, for the coupled nonlinear linear lattice we find both light and dark Solitons but no lattice compression [1, 121]. No domain wall solitons in the higher dimensions.

7.4 ONE DIMENSIONAL LATTICE

Using the parameters of the double well, A and B, identified in section II, we write the equation of motion [87a] for the lattice as

$$m\ddot{\phi} - |A|\phi + B\phi^3 - mc_0{}^2\phi'' = 0 \tag{3}$$

Following [80] we put $\phi = f(x - vt)$ and we obtain

$$m(v^2 - c_0^2)f'' + Af + Bf^3 = 0 \tag{4}$$

Introduce the dimensionless variables

$$m(c_0^2 - v^2)/|A| = \xi^2 \text{(Length squared)} \tag{5}$$

$$\frac{f}{u_0} = \eta \tag{6}$$

$$\frac{(x - vt)}{\xi} = s \tag{7}$$

The dimensionless form of the equation is

$$\frac{d^2\eta}{ds^2} + \eta - \eta^3 = 0 \tag{8}$$

As shown in [80] this equation admits Soliton solutions of the form

$$\eta = \tanh(s\sqrt{2}) \tag{9}$$

which is also the domain wall solution. Note that the tanh solutions are domain walls which have been observed in nonlinear optical lattices [79]. Domain wall Solitons induce lattice compression as observed in [104].

7.5 TWO DIMENSIONAL LATTICE

To develop the equations for the coupled lattice we note first that a photo refractive lattice with two different types of atoms with an exponential interaction [90] can approximate the interaction between the atoms. One may then write the Lagrangian for the coupled lattice as

$$\begin{aligned} L &= \int_{-\infty}^{+\infty} \left[2\mu\phi_1^2 - \left(\frac{d\phi_1}{dx}\right)^2 + [\cos(2x) - g_0]\phi_1^4 \right] dx \\ &+ \int_{-\infty}^{+\infty} \left[2\mu\phi_2^2 - \left(\frac{d\phi_2}{dx}\right)^2 + [\cos(2x) - g_0]\phi_2^4 \right] dx \\ &+ \int_{-\infty}^{+\infty} \exp\left(-\frac{A(\phi_1)}{}\phi_1^2 \frac{B(\phi_2)}{}\phi_2^2 \right) dx \end{aligned} \tag{10}$$

Expanding the exponential and using the Euler Lagrange equations we get

$$m_1\ddot{\phi}_1 + A_1\phi_1 + B_1\phi_1^3 - mc_1^2\phi_1^{"} + VA\ (\phi_1)B\ (\phi_2)\phi_1\phi_2^2 = 0 \tag{11}$$

$$m_2\ddot{\phi}_2 + A_2\phi_2 + B_2\phi_2^3 - mc_2^2\phi_2^{"} + VA\ (\phi_1)B\ (\phi_2)\phi_2\phi_1^2 = 0 \tag{12}$$

$$A_1 = 2\mu_1, A_2 = 2\mu_2 \tag{13}$$

$$B_1 = Cos(2x) - g_{01} \tag{14}$$

$$B_2 = Cos(2x) - g_{02} \tag{15}$$

We look for traveling wave solutions of the form

$$\phi_1 = f_1(z - v_1 t)\ , \tag{16}$$

$$\phi_2 = f_2(z - v_2 t) \tag{17}$$

Here v_1, v_2 are the velocities of the Soliton waves in the two lattices

$$m_1\left(v_1^2 - c_1^2\right) f_1^{"} + A_1 f_1 + B_1 f_1^3 + V_1 A\ (\phi_1) B\ (\phi_2)\ f_2^2 = 0 \tag{18}$$

$$m_2\left(v_2^2 - c_1^2\right) f_2^{"} + A_1 f_2 + B_1 f_2^3 + VA\ (\phi_1) B\ (\phi_2)\ f_2 f_1^2 = 0 \tag{19}$$

We convert the above coupled equations into the dimensionless form

Let $$\frac{m_1\left(c_1^2 - v_1^2\right)}{|A_1|} = \xi_1^2\ \frac{f}{u_{01}} = \eta_1 \tag{20}$$

$$\frac{(x - v_1 t)}{\xi_1} = s_1 \tag{21}$$

And $$\frac{m_2\left(c_2^2 - v_2^2\right)}{|A_2|} = \xi_2^2\ \frac{f}{u_{02}} = \eta_2 \tag{22}$$

$$\frac{(x - v_2 t)}{\xi_2} = s_2 \tag{23}$$

Using these above equations we obtain (Assuming ($AB = B_1 = B_2$ or

$B_1 = B_2\ implies\ g_{01} = g_{02}$) we obtain

$$\frac{d^2\eta_1}{ds^2} + \eta_1 - \eta_1^3 + \eta_1\eta_2^2 = 0 \tag{24}$$

$$\frac{d^2\eta_2}{ds^2} + \eta_2 - \eta_2^3 + \eta_1^2\eta_2 = 0 \tag{25}$$

Adding the two equations one obtains

$$\frac{d^2}{ds^2}\left(\eta_1 + \eta_2\right) + \left(\eta_1 + \eta_2\right) - \left(\eta_1 + \eta_2\right)^3 + 4\eta_1\eta_2\left(\eta_1 + \eta_2\right) = 0 \tag{26}$$

We put $$\eta_1 + \eta_2 = y, \eta_1 = ke^{i\theta}, \eta_2 = \frac{1}{4k} e^{-i\theta} \tag{27}$$

$$\frac{d^2y}{ds^2}+y-y^3+y=0 \tag{28}$$

$$\frac{d^2y}{ds^2}+2y=y^3 \tag{29}$$

The solution to the above equation can be obtained via elliptic equations using the method outlined in [80] $\eta_1+\eta_2=\tanh(\sigma)$ (30)

Or $ke^{i(\pm 2n\pi+\theta)}+\frac{ke^{-i(\pm 2n\pi+\theta)}}{4k}=\tanh(\sigma)$ (31)

Taking the real part we obtain $k\cos(\pm 2n\pi\theta)+\frac{\cos(\pm 2n\pi\theta)}{4k}=\tanh(\sigma)$ (32)

$$\cos(\pm 2n\pi+\theta)=\frac{\tanh(\sigma)}{(4k^2+1)} \tag{33}$$

Since $(\pm 2n\pi+\theta)\neq 0$ we obtain $4k^2=1 \, or \, k=\pm\frac{1}{2}$ (34)

Hence $\cos(\pm 2n\pi+\theta)=\frac{1}{2}\tanh(\sigma)$ (35)

$$\theta=\cos^{-1}\left(\frac{1}{2}\tanh(\sigma)\right)\pm 2n\pi \tag{36}$$

$\pm n$ is called the winding number.

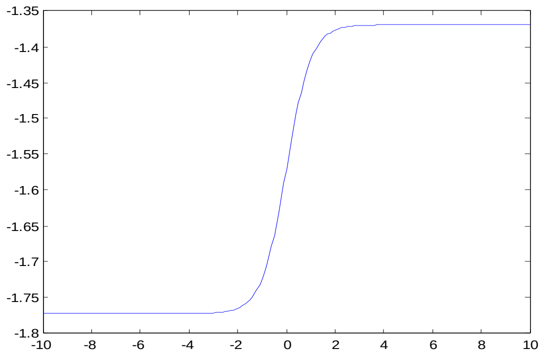

Figure. 4(a) Phase angle θ vs the argument of the tanh function

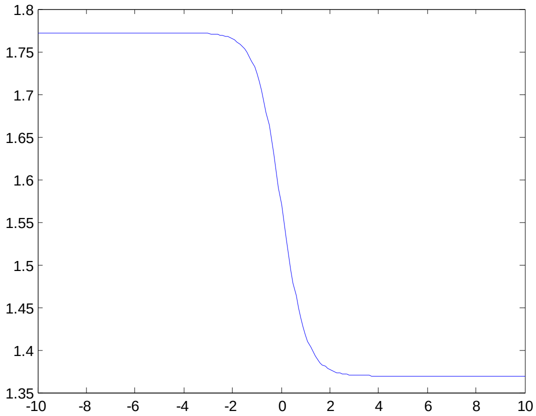

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Figure 4 (b).Phase angle θ vs the argument of the tanh function

In both Figure.4 (a) and Figure. 4 (b) we plot the phase angle θ vs. the argument of the tanh function. The phases are opposite. These correspond to the light and dark Solitons which have been observed in nonlinear optical lattices [102, 27]. These equations actually describe matter wave oscillations (with opposite phases) in the system. In this picture two wells of the double well of the lattice vibrate in opposite phases as observed experimentally in [35].The experimental justification of the solutions given here also implies that exponential approximation of the interaction and the assumption $g_{01} = g_{02}$ given earlier is correct.

7.6 TUNNELING

Recently photon assisted tunneling has been observed in optical lattices [115] subject to a sinusoidal shaking of the lattice. To a first approximation we have harmonic oscillator states in each well. The probability of tunneling through a potential of height V and width a is given by (in the WKB approximation)

$$P = \frac{1}{1+\dfrac{V^2 Sinh^2\left(\dfrac{\sqrt{2m\left(V-E\right)a^2}}{\hbar}\right)}{4E\left(V-E\right)}} \qquad (37)$$

where E is energy of a state in the harmonic oscillator well and is given by [3]

$$\varepsilon_{0,s} = \frac{1}{2}\left(\frac{2A}{m^*}\right)^{1/2}\left\{1 \pm \frac{1}{2}\exp\left[-u_0\left(\frac{A^2}{B}m^*\right)^{1/2}\right]\right\} \tag{38}$$

Where u_0 is the potential minimua and m^* is the effective mass. . For resonance to occur the energy of the tunneling particles must be at least equal to

the height of the double well hump or ($-\frac{1}{4}|A|^2 / B$). This means

$$\frac{\hbar^2 k^2}{2m^*} = \frac{A^2}{4B} \tag{39}$$

Using $m^* = \hbar = 1$, we obtain $k = \left(\frac{A^2}{2B}\right)^{1/2}$ (40)

Oscillation of the lattice means that the minima of the double well are undergoing an oscillation. This means $u_0 = ACos(kx - wt)$ (41)

Coherent tunneling implies that in the same phase tunneling can occur. This means tunneling can occur both at the start and end of the oscillation. One complete

oscillation is defined by $k\Delta x = 2\pi$ or $k = \frac{2\pi}{\Delta x}$ (42)

Taking $\Delta x = u_0$ we obtain $k = \frac{2\pi}{u_0}$. (43)

Taking into account the fluctuation in u_0

we write the sinh^2 term as $S\text{in}\,h^2\left[\sqrt{2m\left(V - \frac{1}{2}\left(\frac{2A}{m^*}\right)^{1/2}\left\{1 \pm \left[\exp\left(-u_0\left(\frac{A^2 m^*}{2B}\right)^{1/2} + \left(\left(\frac{A^2 m^*}{2B}\right)^{1/2}\frac{1}{u_0}\right)\right)\right]\right\}\right)}\right]$ (44)

Using the expansion

$$e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n \tag{45}$$

We obtain

$$S\text{in}\,h^2\left[\sqrt{2m\left(V - \frac{1}{2}\left(\frac{2A}{m^*}\right)^{1/2}\left\{1 \pm \frac{1}{2}\sum_{n=-\alpha}^{n=+\alpha}\left(J_n\left(\frac{A^2}{2B}m^*\right)^{1/2} + J_n\left(\frac{-A^2}{2B}m^*\right)^{1/2}\right)\right\}\right)}\right] \tag{46}$$

Using this expression we simulated the probability of transmission. The results are shown in Figure 5

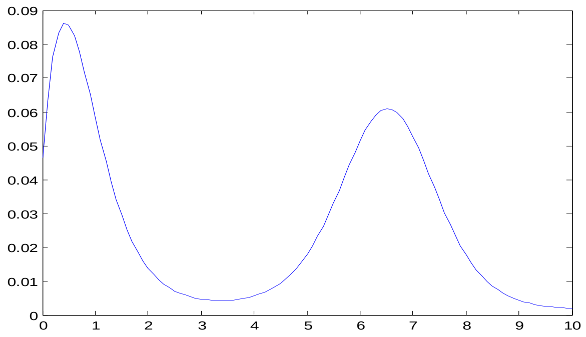

Figure 5 Transmission probability vs energy

The simulated profile agrees favorably with the experimental results of [121,104]. We conclude that the double well model provides a reasonable basis for the study of the various properties of the nonlinear optical lattice.