CHAPTER 6

LONG WAVE LENGTH TANH SOLITON SOLUTIONS OF SINE GORDON EQUATION

6.1 INTRODUCTION

In this chapter we find the long wave Length Soliton Solution of Sine Gordon Equation. Recently H. Sakaguchi and B.A Malomed proposed a novel technique for finding the long wavelength solutions of the Gross Pitaevskii equation. We have extended their technique to Sine Gordon Equation. We find the Greens function corresponding to the Sine Gordon equation in the long wave length limit. The derived Green's function agrees with that obtained for the Josephson's junction intuitively. Then we find the change in probability distribution as we move across the Josephson junction.

The Sine Gordon equation which we have taken to solve is $\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}+\sin\psi=0$

Sine Gordon Equation is a partial differential equation which appears in differential geometry and relativistic field theory. The equation, as well as several solution techniques, was known in the 19th century, but the equation grew greatly in importance when it was realized that it led to solutions ("kink" and "antikink") with the collisional properties of solitons. The sine-Gordon equation also appears in a number of other physical applications, including the propagation of fluxons in Josephson junctions (a junction between two superconductors), the motion of rigid pendula attached to a stretched wire, and dislocations in crystals.

Recently H. Sakaguchi and B. A. Malomed in their seminal paper [111] proposed a novel expansion technique (2) to obtain the long wavelength Soliton solutions for the Gross-Pitavskii equation. By substituting the expansion (2) in the nonlinear differential equation we obtain (by comparing coefficients) the effective equations in the long wavelength region. The effective equation is put in the form of a conservation law (4). The spatial component of the conservation law is an eigenvalue equation. However for Sine Gordon, variable transformations are required to obtain the corresponding Schrodinger equation. The formalism so obtained is applied to the Josephson junction.

6.2 SINE GORDON EQUATIONIN THE LONG WAVE LENGTH LIMIT

The Sine –Gordon equation is [105]

$$\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}+\sin\psi=0 \tag{1}$$

We look for solutions of the form [111a]

$$\psi\left(x,t\right)=\psi^{(0)}\left(x,t\right)+\psi^{(1)}\left(x,t\right)\cos\left(2x\right)+.... \tag{2}$$

where $\psi^{(0)}(x,t)$ and $\psi^{(1)}(x,t)$ are slowly varying functions of x and t in comparison to cos(2x) and $\psi^{(1)}_{xx}(x,t)<<\psi^{(1)}(x,t)$. (3)

Using the expansion [21]

$$e^{ixs\mathrm{i}\,\mathrm{n}\,\phi}=\sum_{n=-\infty}^{n=+\infty}J_n\left(x\right)e^{in\phi} \tag{4}$$

Where $J_n(x)$ is the Bessel function.

$$\sin\psi=\sin\left(\psi^{(0)}\left(x,t\right)+\psi^{(1)}\left(x,t\right)\cos\left(2x\right)\right) \tag{5}$$

$$\cos\left(x\sin\phi\right)=\sum_{n=-\infty}^{\infty}J_n(x)\cos n\phi \tag{6}$$

$$\sin\left(x\sin\phi\right)=\sum_{n=-\infty}^{\infty}J_n(x)\sin n\phi \tag{7}$$

$$\cos\left(\psi^{(1)}\left(x,t\right)\cos\left(2x\right)\right)=\sum_{n=-\infty}^{\infty}J_n(\psi^{(1)}\left(x,t\right))\cos n\left(2x+\frac{\pi}{2}\right) \tag{8}$$

$$\sin\left(\psi^{(1)}\left(x,t\right)\cos\left(2x\right)\right)=\sum_{n=-\infty}^{\infty}J_n\left(\psi^{(1)}\left(x,t\right)\right)\sin n\left(2x+\frac{\pi}{2}\right) \tag{9}$$

Equating the coefficients of cos2x

$$\psi^{(1)}_{tt}-\psi^{(1)}_{xx}+4\psi^{(1)}+\cos\left(\psi^{(0)}\right)J_1\left(\psi^{(1)}\right)=0 \tag{10}$$

$$J_1\left(\psi^{(1)}\right)=\frac{\psi^{(1)}}{2}-\frac{\left(\psi^{(1)}\right)^3}{16} \tag{11}$$

Collecting the terms sub (11) in (10) one obtain

$$\psi_{tt}^{(1)}-\psi_{xx}^{(1)}+A\psi^{(1)}+B\left(\psi^{(1)}\right)^3=0 \tag{12}$$

where

$$A=-4.5, B=0.0625 \tag{13}$$

Note that (13) conforms to the condition for a double well potential and thereby the existence of tanh solitons [80].

Let us look for travelling wave solutions of the sine-Gordon equation (5.1) of the form $\psi^{(1)}=f(x-vt)$ (14)

Using (14) in (12) we get

$$\left(v^2-1\right)f''+Af+Bf^3=0 \tag{15}$$

This equation has been studied by J.A Krumhansl and J.R Schrieffer [80] who find tanh Soliton solution

$$\psi(x,t)=\tanh\left(\frac{x-vt}{\sqrt{2}\zeta}\right) \tag{16}$$

$$\zeta=\frac{1-v^2}{|A|} \tag{17}$$

Equation (17) represents a localized solitary wave, called kink soliton solution.

We note that tanh soliton solution have been found in Josephson's junction (in the long wavelength limit) which are governed by the Sine Gordon equation. Such domain wall soliton solutions have been observed in switching experiments using Nb/Ru/Sr_2RuO_4 junctions [86,105].

6.3. CONSERVATION EQUATION.

We now derive the conservation equation [15, 54, 92, 81] corresponding to (6). Define ρ such that

$$\psi_t^{(1)} = \rho, \psi^{(1)} = \frac{\partial \rho}{\partial x} \tag{18}$$

Using these substitutions in (1) we obtain

$$\rho_t = \rho_{xxx} - A\frac{\partial \rho}{\partial x} + B\left(\frac{\partial \rho}{\partial x}\right)^3 \tag{19}$$

This can be written as a conservation equation

$$\rho_t = \left(\rho_{xx} - A\rho + \frac{B}{4}\left(\frac{\partial \rho}{\partial x}\right)^4\right)_x = 0 \tag{20}$$

This gives the equation

$$\rho_{xx} - A\rho + \frac{B}{4}\left(\frac{\partial \rho}{\partial x}\right)^4 = C \tag{21}$$

Where c is a constant, we take c=0

6.4. APPLICATION TO JOSEPHSON JUNCTIONS.

Josephson junction consists of two super conductors separated by a thin oxide layer. Each super conductor is characterized by a wave function ψ_1, ψ_2 and phase given by the expressions $\psi_1 = \sqrt{n}e^{i\theta_1}$ (22)

$$\psi_2 = \sqrt{n}e^{i\theta_2} \tag{23}$$

Josephson's phase is defined as $\psi^{(1)} = \theta_2 - \theta_1$ (24)

Josephson's phase satisfies the Sine-Gordon equation (1). In the long wavelength limit the Josephson's phase satisfies (5). Josephson's phase is a function of both space and time. The spatial integral defined as

$$\rho = \int \psi^{(1)}(x,t)dx \tag{25}$$

The spatial integral of Josephson's phase is a conserved quantity as derived in (12). Further ρ satisfies the eigenvalue equation (15). We use the boundary condition $\rho \to 0$ as $x \to \pm L$. We use the trial solution

$$\rho(x) = Ce^{-kx} \tag{26}$$

$$k^2C - AC - \frac{Bc^4k^4e^{-3kL}}{4} = 0 \tag{27}$$

$$k^2C - AC - \frac{Bc^4k^4e^{3kL}}{4} = 0 \tag{28}$$

Subtracting (26) from (25) we obtain

$$\frac{Bc^4k^4}{2} sinh(3kL) = 0 \tag{29}$$

This implies $3kL = -in\pi$ (30)

Since L is real k must be imaginary. In other words (25) represents an oscillatory solution. Let

$$k = -i\kappa \tag{31}$$

Where κ is a real quantity. Substituting (30) in (29) we get

$$\kappa = \frac{n\pi}{3L} \tag{32}$$

This defines wave-vectors in (26). Note that only certain wave vectors are permissible. This is expected, as we are in the quantum domain where only discrete states are allowed.

6.5. CONCLUSION.

We found the long wave length kink soliton solution of Sine Gordon equation. We have derived the conservation equation of Sine Gordon equation in the long wavelength limit. The kink solution in this limit as well as conservation laws in this limit are derived. The results are applied to the Josephson's junction and we found Josephson's phase exhibits spatial sinusoidal oscillations. Further we find that the spatial integral of the Josephson's phase is a conserved quantity.