CHAPTER 4

LONG WAVE LENGTH SOLITON SOLUTIONS OF NAVIER STOKES EQUATION

4.1 INTRODUCTION

In this chapter we find the long wavelength Soliton solution of Navier Stokes equation. We have used Becker's field theoretic formulation of Navier Stokes equation to derive the equation of motion for a double well potential [14,72,34,122,125,43,13,83]. We find Domain wall solutions. Double well profiles are found in Cahn-Hillard Navier Stokes system. We suggest that Domain walls exist in the Cahn-Hillard system. To investigate this further we compute the Free Energy of the Domain Walls. The plot of the Domain Wall Free Energy agrees very well the profiles computed by other authors. Finally we use the method of Sakaguchi and Malomed to obtain the long wavelength expansion of the Navier Stokes equation. In this limit we find conservation laws. This result implies that under certain conditions the infinite array of conservation laws is also applicable to the Navier Stokes equation.

4.2 NAVIER STOKES EQUATION

Navier–Stokes equations [51]named after Claude-Louis Navier and George Gabriel Stokes, describe the motion of fluid substances. These equations arise from applying Newton's second law to fluid motion, together with the assumption that the fluid stress is the sum of a diffusingviscous term (proportional to the gradient of velocity) and a pressure term - hence describing viscous flow.

The equations are useful because they describe the physics of many things of academic and economic interest. They may be used to model the weather, ocean currents, water flow in a pipe and air flow around a wing. The Navier–Stokes equations in their full and simplified forms help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution, and many other things. Coupled with Maxwell's equations they can be used to model and study magnetohydrodynamics.

The Navier–Stokes equations are also of great interest in a purely mathematical sense. Somewhat surprisingly, given their wide range of practical uses, mathematicians have not yet proven that in three dimensions solutions always exist (existence), or that if they do exist, then they do not contain any singularity (smoothness). These are called the Navier–Stokes existence and smoothness problems. The Clay Mathematics Institute has called this one of the seven most important open problems in mathematics and has offered a US$1,000,000 prize for a solution or a counter-example.

4.3 NONLINEARITY

The Navier–Stokes equations are nonlinear partial differential equations in almost every real situation. In some cases, such as one-dimensional flow and Stokes flow (or creeping flow), the equations can be simplified to linear equations. The nonlinearity makes most problems difficult or impossible to solve and is the main contributor to the turbulence that the equations model.

The nonlinearity is due to convective acceleration, which is an acceleration associated with the change in velocity over position. Hence, any convective flow, whether turbulent or not, will involve nonlinearity. An example of convective but laminar (nonturbulent) flow would be the passage of a viscous fluid (for example, oil) through a small converging nozzle. Such flows, whether exactly solvable or not, can often be thoroughly studied and understood.

4.4 TURBULENCE

Turbulence is the time dependent chaotic behavior seen in many fluid flows. It is generally believed that it is due to the inertia of the fluid as a whole: the culmination of time dependent and convective acceleration; hence flows where inertial effects are small tend to be laminar (the Reynolds number quantifies how much the flow is affected by inertia). It is believed, though not known with certainty, that the Navier–Stokes equations describe turbulence properly. The numerical solution of the Navier–Stokes equations for turbulent flow is extremely difficult, and due to the significantly different mixing-length scales that are involved in turbulent flow, the stable solution of this requires such a fine mesh resolution that the computational time becomes significantly infeasible for calculation (see Direct numerical simulation). Attempts to solve turbulent flow using a laminar solver typically result in a time-unsteady solution, which fails to converge appropriately. To counter this, time-averaged equations such as the Reynolds-averaged Navier–Stokes equations (RANS), supplemented with turbulence models, are used in practical computational fluid dynamics (CFD) applications when modeling turbulent flows. Some models include the Spalart-Allmaras, k-ω (k-omega), k-ε (k-epsilon), and SST models which add a variety of additional equations to bring closure to the RANS equations. Another technique for solving numerically the Navier–Stokes equation is the Large eddy simulation (LES). This approach is computationally more expensive than the RANS method (in time and computer memory), but produces better results since the larger turbulent scales are explicitly resolved.

Recently R. J. Becker [17] has developed the Lagrangian approach to Navier Stokes equation. In Becker's formalism the velocity vector is defined in terms of the gradient of a scalar field ψ and curl of a vector field $\vec{A}$ as

$$\vec{v} = \nabla \psi + \nabla X \vec{A} \tag{1}$$

We consider a system where the fluid has a low velocity and a stream lined flow. Here we can safely assume

$$\vec{A} = 0 \tag{2}$$

The Lagrangian density for the Navier Stokes equation is then given according to the formalism of Becker [17] by,

$$L = L_0 + L_I \tag{3}$$

$$L_0 = \dot{\psi}^2 + \frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 \tag{4}$$

where we have neglected the Diffusion term. The interaction term is taken as the Double well potential $$L_I = -A\frac{\psi^2}{2} - B\frac{\psi^4}{4} \quad (5)$$

Double well potentials admit Soliton solutions. Solitons have been both observed in fluids [33] as well as predicted via calculations and simulations. From the Euler Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) + \frac{d}{dx}\left(\frac{\partial L}{\partial \psi_x}\right) - \frac{\partial L}{\partial \psi} = 0 \quad (6)$$

Note that a double well potential admits bound states as has indeed been found in the long wavelength limit. One can derive equation of motion

$$\ddot{\psi} + A\psi + B\psi^3 - \psi'' = 0 \quad (7)$$

whose solution is as obtained by [80] is $$\psi = \tanh\left(\frac{x - vt}{\sqrt{2}}\right) \quad (8)$$

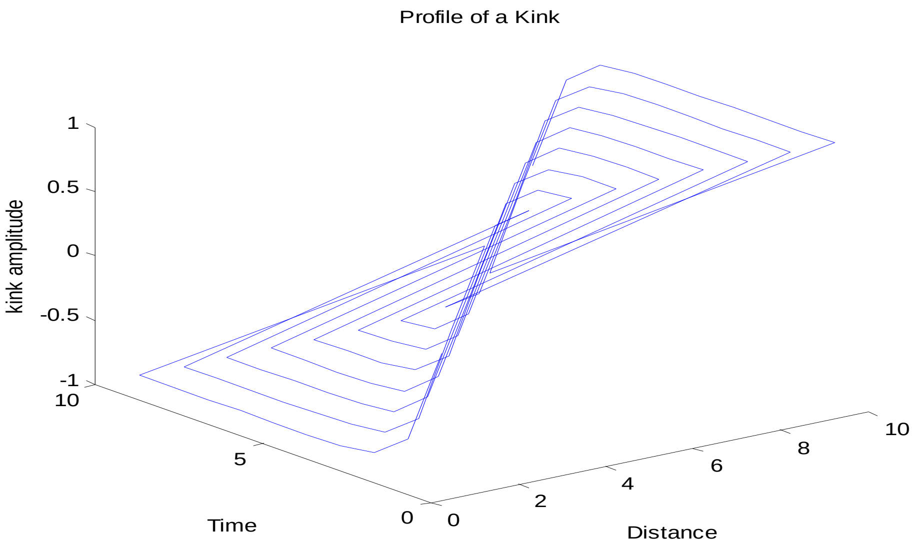

Figure 1 Profile of Kink

This is also known as the domain wall solution (see Figure.1). Shapes in the form of such domains are seen near the sea shore when Tsunami occurs. In a sense this model is unrealistic as it discusses fluid flow in one dimension only. However a 3 dimensional model of this idea is found in the Cahn-Hillard Navier Stokes model.

Here one finds a double well model in 3 dimensions. It is also of interest the thermodynamic equilibrium of such a one dimensional model as the Free Energy of the Cahn-Hillard has been simulated experimentally [23].

4.5 EQUILIBRIUM BEHAVIOR

We are interested in the thermodynamic equilibrium of the Navier Stokes system. The functional integral form of the Partition function is given by

$$Z = \int [\delta u \delta p] e^{-\beta H(p,u)} \tag{9}$$

Scalpino, Sears and Ferrel [112] (SSF), Kac and Helfand [70] have shown how to evaluate (9). In particular Krumhansl and Schrieffer [80a] have computed the Free Energy of Domain walls in the double well model. A plot of this result is shown in Figure 2 One sees that the Free energy plots in Figure. 2 are identical with the Free Energy plots of F.Boyer, C. Lapuerta, S. Minjeoud, B. Piar and M. Quintard [23] obtained after extensive simulation. This implies that double well model we have used is indeed correct.

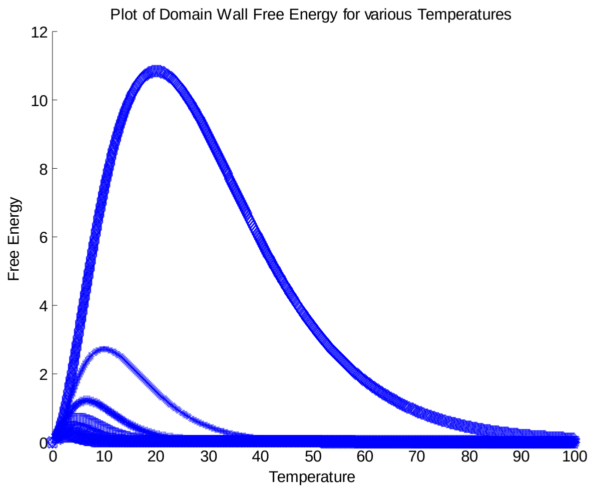

Figure 2 Domain Wall Free Energy

4.6 LONG WAVE LENGTH SOLUTIONS

H. Sakaguchi and B. A. Malomed in their seminal paper [111] proposed a novel expansion technique (11) to obtain the long wavelength Soliton solutions for the Gross-Pitavskii equation. By substituting the expansion (10) in the Navier Stokes equation we obtain (by comparing coefficients) the effective equations in the long wavelength region. The effective equation is put in the form of a conservation law (15). The spatial component of the conservation law is an eigenvalue equation. For the KdV equation the eigenvalue equation is the Schrodinger equation. However for Navier Stokes some variable transformations are required to obtain the corresponding Schrodinger equation. The important point is that in the asymptotic limit the nonlinear differential equation becomes equivalent to the Schrodinger equation. In each case the Schrodinger equation is solved and the Green's function obtained from the eigen-functions.

4.7 NAVIER STOKES EQUATION IN LONG WAVELENGTH.

Here again we concentrate on the scalar field whose gradient gives the velocity vector. The equation of the scalar potential are

$$\ddot{\psi} - D\nabla^2\dot{\psi} - c^2\nabla^2\psi = 0 \tag{10}$$

This equation reflects the interplay between diffusion and propagation.We look for solutions of the form $\psi(x,t) = \psi^{(0)}(x,t) + \psi^{(1)}(x,t)\cos(2x) + \ldots.$ (11)

Equating the coefficients of cos(2x) we get

$$\psi_{tt}^{(1)} = D\left[\psi_{txx}^{(1)} - 4\psi_t^{(1)}\right] + c^2\left[\psi_{xx}^{(1)} - 4\psi^{(1)}\right] \tag{12}$$

Navier Stokes Equation $\psi_{tt}^{(1)} + 4D\psi_t^{(1)} = D\psi_{txx}^{(1)} + c^2\psi_{xx}^{(1)} - 4c^2\psi^{(1)}$ (13)

Define $\psi_t^{(1)} + D\psi_t^{(1)} = \xi$ (14)

The conservation equation is $\xi_t + \left(\left(c^2 - Dv\right)\psi_{xx} - 4c^2\psi\right)_x = 0$ (15)

$$\left(c^2 - Dv\right)\psi_{xx} = \left(4c^2\right)\psi \tag{16}$$

OR $\psi_{xx} = \dfrac{\left(4c^2\right)\psi}{\left(c^2 - Dv\right)}$

Where $k^2 = \dfrac{\left(4c^2\right)}{\left(c^2 - Dv\right)}$ (17)

The above equation is an eigen-value equation. For $Dv > c^2$ the solution is

$$\psi \sim \sin(kx + \delta) \tag{18}$$

This corresponds to the bound state. Bound states in Navier Stokes equations have been reported [23]. For $Dv < c^2$ the solution is $\psi = Ae^{kx} + Be^{-kx}$ (19)

For slowly diffusing media if the velocity is larger than the speed of sound in the media then the disturbance will propagate as sinusoidal waves. In the opposite limit the disturbance will propagate as decaying exponentials.

4.8 CONCLUSION.

We have used Becker's field theoretic formulation of Navier Stokes equations to obtain the Lagrangian corresponding to a double well potential. The double well potential has domain wall solutions. The profile of the Free Energy for Domains (as computed by Krumhansl and Schreiffer) is in good agreement with Free Energy profile as computed by [20] via computer simulations. Thus the Cahn-Hillard model for two or three phase flow thus can be accounted for Domain wall solutions. Further we have used the formulation of Sakaguchi and Malomed [111] to derive the long wavelength approximation to the Navier Stokes equation. In the long wavelength approximation we have derived the conservation law. This result implies that only in the long wavelength limit the array of infinite conservation laws are applicable to the Navier Stokes equation.