

M.Sc. MATHEMATICS
FIRST SEMESTER
REAL ANALYSIS
MSM – 101
[USE OMR FOR OBJECTIVE PART]

2025/12

**SET
A**

Duration: 3 hrs.

Full Marks: 70

Time: 30 min.

(Objective)

Marks: 20

1×20=20

Choose the correct answer from the following:

- The power series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ is
 - nowhere convergent
 - everywhere convergent
 - convergent at $x = 1$ only
 - convergent at $x = \frac{1}{2}$
- The radius of convergence $\mathbb{R}$ for a power series $\sum_{n=0}^{\infty} a_n x^n$ is computed by the formula:
 - $\lim_{n \rightarrow \infty} |a_n|$
 - $\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$
 - $\frac{1}{\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}}$
 - None of these
- If a power series $\sum a_n x^n$ converges for $x = x_0$, then it is absolutely convergent for $x = x_1$ where
 - $x_1 < x_0$
 - $|x_1| < |x_0|$
 - $x_1 < |x_0|$
 - $|x_1| < x_0$
- The series $\sum \frac{x^n}{n^2}$ is uniformly convergent in
 - $[-1, 1]$
 - $(-1, 1]$
 - $(-1, 1)$
 - $[-1, 1)$
- The radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ is
 - 0
 - ∞
 - 1
 - None of these
- The radius of convergence of the series $1 + 2x + 3x^2 + 4x^3 + \dots$ is
 - $\frac{1}{2}$
 - 1

c. $\frac{3}{2}$

d. None of these

7. Consider $\mathbb{R}$, the set of real numbers with the metric $d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Then $d(1, 3)$ is equal

a. 0
c. ∞

b. 1
d. $\frac{1}{2}$

8. Consider $\mathbb{R}^2 = \{p : p = (x, y), x, y \in \mathbb{R}\}$. Define $d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$d(p, q) = |x_1 - x_2| + |y_1 - y_2|, \text{ where } p = (x_1, y_1), q = (x_2, y_2)$$

If $p = (-1, 1), q = (2, -1)$ then $d(p, q)$ is

a. 4
c. 6

b. 5
d. 7

9. Define $d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$d(x, y) = \text{Max}\{|x_1 - y_1|, |x_2 - y_2|\}, \text{ where } x = (x_1, x_2), y = (y_1, y_2)$$

If $x = (2, 1), y = (-3, 2)$ then $d(x, y)$ is

a. 0
c. 5

b. 2
d. 1

10. For any two points $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ define $d(x, y) = |x_1 - y_1|$. Then

a. $d(x, y) = 0 \Rightarrow x = y$
c. $d(x, y) = 0 \Leftrightarrow x = y$

b. $x = y \Rightarrow d(x, y) = 0$
d. None of these

11. A series of functions $\sum f_n$ will converge uniformly on $[a, b]$ if there is a convergent series $\sum M_n$ of positive numbers such that for all $x \in [a, b]$ and for all n

a. $|f_n(x)| < M_n$
c. $|f_n(x)| \leq M_n$

b. $|f_n(x)| > M_n$
d. $|f_n(x)| \geq M_n$

12. Show that the sequence $\langle f_n \rangle$, where $f_n(x) = x^n$ is uniformly convergent to the limit function $f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$ in the interval

a. $[0, 1]$
c. $[0, k], k > 1$

b. $[0, k], k < 1$
d. $[-1, 1]$

13. $\sum \frac{\cos n\theta}{n^p}$ is uniformly convergent for all real values of θ for

a. $p \leq 1$
c. $p > 1$

b. $p \geq 1$
d. $p < 1$

14. Let a sequence $\{f_n\}$ of real functions converges uniformly to a real function f so that given $\epsilon > 0$, there exists a positive integer m so that $|f_n(x) - f(x)| < \epsilon, \forall n \geq m$ for $x \in [a, b]$. Then

a. m depends on $x \in [a, b]$ and not on ϵ
c. m is independent of both ϵ and $x \in [a, b]$

b. m depends on ϵ and not on $x \in [a, b]$
d. None of these

15. Consider the series $\sum f_n$ of functions where $f_n(x) = \frac{x^2}{(1+x^2)^n}$, $x \in \mathbb{R}$. The series converges to a sum function f given by
- $f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$
 - $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$
 - $f(x) = 0, x \in \mathbb{R}$
 - $f(x) = 1, x \in \mathbb{R}$
16. For any interval $[a, b]$ in $\mathbb{R}$ the length of $[a, b]$ is
- $a + b$
 - $a - b$
 - $b - a$
 - None of the above
17. The power series $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ is
- Convergent at $x = 0$ only.
 - Everywhere Convergent
 - Nowhere Convergent
 - None of these
18. The interval of convergence of the power series $1 + x^2 + x^4 + x^6 + \dots$ is
- $-1 \leq x < 1$
 - $-1 < x \leq 1$
 - $-1 < x < 1$
 - $-1 \leq x \leq 1$
19. The radius of convergence of the power series $x + \frac{x^2}{2!} + \frac{2!}{3!}x^3 + \frac{3!}{4!}x^4 + \dots$
- $\frac{1}{e}$
 - e
 - $1 + e$
 - $1 - e$
20. Consider the discrete metric space $(\mathbb{R}, d)$, where $\mathbb{R}$ denote the set of real numbers and

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Then $d(1, -1)$ is

- 2
- 0
- 1
- 1

(Descriptive)

Time : 2 hrs. 30 min.

Marks : 50

[Answer question no.1 & any four (4) from the rest]

- Explain what you mean by pointwise convergence of a sequence of functions $\{f_n(x)\}$ define on $[a, b]$. Examine if the sequence of functions $\{f_n(x)\}$, $f_n(x) = x^n$ is convergent on $[0, 1]$ or not. Find the limit function $f(x)$ of the sequence if it is convergent. 3+5+2
=10
- What are partial sums of a series of function $\sum f_n(x)$? How do you define the pointwise convergence of a series of functions $\sum f_n(x)$? 2+3+5
=10
Consider the series $\sum f_n(x)$ where $f_n(x) = \frac{x^2}{(1+x^2)^n}$, $x \in \mathbb{R}$. Show that $\sum f_n(x)$ converges pointwise for all $x \in \mathbb{R}$ to a sum function given by

$$f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

3. Explain the concept of uniform convergence of a sequence of function $\{f_n(x)\}$ as well as a series of function $\sum f_n(x)$ on an interval $[a, b]$. 2+2+3+3
=10

Test for uniform convergence of the sequence of functions $\{f_n(x)\}$ where $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$. Show that the sequence does not converge uniformly in an interval $[a, b]$ containing 0

4. a. What is a power series in x ? Explain the following terms with examples: 2+3+5
=10

- (i) Radius of convergence of a power series.
- (ii) Nowhere convergent power series.
- (iii) Everywhere convergent power series

b. Find the radius of convergence of the power series

$$\frac{1}{2}x + \frac{1}{2} \cdot \frac{3}{5}x^2 + \frac{1}{2} \cdot \frac{3}{5} \cdot \frac{5}{8}x^3 + \dots$$

5. a. If a power series $\sum a_n x^n$ converges for $x = x_0$ then prove that it is absolutely convergent for every $x = x_1$ such that $|x_1| < |x_0|$. 5+5=10

b. Show that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots, -1 \leq x < 1$$

and hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots$

6. a. Define a metric space (X, d) . Explain the concepts of 2+3+2+3
=10
- (i) discrete metric on $\mathbb{R}$
 - (ii) usual metric on $\mathbb{R}$
 - (iii) usual metric on $\mathbb{R}^2$

b. What is a power series in x ? Explain the following terms with examples-

- (i) Radius of convergence of a power series
- (ii) Nowhere convergent power series
- (iii) Everywhere convergent power series

7. a. Let (X, d) be a metric space. Define $d_1 : X \times X \rightarrow \mathbb{R}$ by

5+5=10

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}, x, y \in X$$

Examine if (X, d_1) is also a metric space or not.

- b. What do you mean by an open sphere $S_r(x_0)$ with centre x_0 and radius r in a metric space (X, d) . Find $S_{\frac{1}{2}}(0)$ in $\mathbb{R}$ with respect to both usual metric and discrete metric.

8. a. State and prove Weierstrass's M-test for uniform convergence of a series of functions $\sum f_n(x)$ in $[a, b]$. 5+5=10

- b. Apply Weierstrass's M-test to show that $\sum x^n \cos n\theta$ converges uniformly for all real values of θ if $0 < x < 1$.

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